1 Measurement Revisited

This linear superposition $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is part of the private world of the qubit. For us to know the state of the qubit, whether this is realized by an electron, a photon, or an electron spin, we must make a measurement. Measuring $|\psi\rangle$ in the standard basis $\{|0\rangle, |1\rangle\}$ yields $|0\rangle$ with probability $|\alpha|^2$, and $|1\rangle$ with probability $|\beta|^2$.

One important aspect of the measurement process is that it alters the state of the quantum system: the effect of the measurement is that the new state is exactly the outcome of the measurement. I.e., if the outcome of the measurement is $|0\rangle$, then following the measurement, the qubit is in state $|0\rangle$. This implies that you cannot collect any additional information about the amplitudes α_j by repeating the measurement on the resuling state. You need to make many identical measurements on a set (ensemble) of equivalent states.

Repeated measurements on a state may however be useful for other reasons. We shall examine this with analysis of the measurement process for photon polarization. The polarization of a photon can be measured by using a polaroid or a calcite crystal. These materials act as filters that select only one component of the electric field vector. See the section on polarization in ScienceTrek at http://www.colorado.edu/physics/2000. A polaroid sheet (suitably oriented) transmits x-polarized photons $|x\rangle$ and absorbs y-polarized photons $|y\rangle$. Thus a photon that is in a superposition $|\phi\rangle = \alpha|x\rangle + \beta|y\rangle$ is transmitted with probability $|\alpha|^2$ if the polaroid sheet is oriented to transmit x and with probability $|\beta|^2$ is the sheet is oriented to transmit y. In the former case the final state is $|x\rangle$, in the latter case it is $|y\rangle$.

Consider passing a photon in state $|\psi\rangle$ through 2 polaroid filters, first an x filter, then a y filter. After the first filter we have $|x\rangle$ with prob. $|\alpha|^2$. After the second filter we have nothing, with prob. 1. Where has the photon gone? During passage through the first filter it was absorbed by the first filter with prob. $|\beta|^2$. If it got through this first filter, it was absorbed by the second filter with prob. 1. Note that the experiment may also be interpreted as the results of identical experiments on many identical photons in state $|\psi\rangle$.

Now consider what happens if we interpose a third polaroid sheet at a 45 degree angle between the first two. Now a photon that is transmitted by the first sheet makes it through the next two with probability 1/4. Why is this? The polarization of light after the first filter is $|x\rangle$. The second filter is oriented at 45 degrees, i.e., it will pass photons with polarization orientation $\vec{v} = \frac{1}{\sqrt{2}}(\vec{x}+\vec{y})$. So let's express $|x\rangle$ in the basis $|v\rangle$ and its orthogonal complement $|v^{\perp}\rangle$. This is also known as the $|+\rangle, |-\rangle$ basis.

$$|x\rangle = \frac{1}{\sqrt{2}}\left(|v\rangle + |v^{\perp}\rangle\right)$$

Now the light passing through the first filter is in state $|x\rangle$ with probability $|\alpha|^2$. The probability this light passes the second filter is equal to the probability that a $|0\rangle$ qubit ends up in $|+\rangle$ when measured in the $|+\rangle, |-\rangle$ basis. Reading off from the above equation, we see that this probability is 1/2. Those photons that do pass successfully through the second filter now have a resulting polarization $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. The probability of this state now passing the third filter oriented in y is then 1/2. What is the overall probability of having a photon pass successfully through all 3 sheets? Answer: $|\alpha|^2/4$, obtained by multiplying the three probabilities for successive passage. The final photon state is $|y\rangle$.

Note that one effect of these measurements is to effectively rotate the plane of polarization of the photon - measurements can thus provide a way to make operations on qubits, although these are not unitary operations.

2 Notation

The notation $\langle v|$ ("bra v") denotes a row vector, the conjugate-transpose of $|v\rangle$, or $|v\rangle^{\dagger}$. For example, $\langle 0| = \left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right)$ and $\langle 1| = \left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)$. More generally,

$$\langle \psi| = \left(\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right)^{\dagger} = \left(\begin{smallmatrix}\bar{\alpha} & \bar{\beta}\end{smallmatrix}\right) = \bar{\alpha}\langle 0| + \bar{\beta}\langle 1| \quad . \tag{1}$$

The Dirac notation can be handy. For example, let

$$|v_1\rangle = a_1|0\rangle + b_1|1\rangle, \quad |v_2\rangle = a_2|0\rangle + b_2|1\rangle \quad . \tag{2}$$

Then $\langle v_1|v_2\rangle$ (shorthand for $\langle v_1|\,|v_2\rangle$) is a matrix product of the 1×2 matrix $\langle v_1|$ and the 2×1 matrix $|v_2\rangle$, or just a scalar:

$$\langle v_1|v_2\rangle = \left(\begin{smallmatrix}\bar{a}_1 & \bar{b}_1\end{smallmatrix}\right)\left(\begin{smallmatrix}a_2 \\ b_2\end{smallmatrix}\right) = \bar{a}_1 a_2 + \bar{b}_1 b_2 \quad . \tag{3}$$

$\langle v_1|v_2\rangle = \overline{\langle v_2|v_1\rangle}$ is an inner product. Note that $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \overline{\langle 1|0\rangle} = 0$. Thus the above equation could have been expanded,

$$\begin{aligned}\langle v_1|v_2\rangle &= (\bar{a}_1\langle 0| + \bar{b}_1\langle 1|)(a_2|0\rangle + b_2|1\rangle) \\ &= \bar{a}_1 a_2\langle 0|0\rangle + \bar{a}_1 b_2\langle 0|1\rangle + \bar{b}_1 a_2\langle 1|0\rangle + \bar{b}_1 b_2\langle 1|1\rangle \\ &= \bar{a}_1 a_2\cdot 1 + \bar{a}_1 b_2\cdot 0 + \bar{b}_1 a_2\cdot 0 + \bar{b}_1 b_2\cdot 1 \\ &= \bar{a}_1 a_2 + \bar{b}_1 b_2 \quad .\end{aligned} \tag{4}$$

In this notation, $\alpha = \langle 0|\psi\rangle$, $\beta = \langle 1|\psi\rangle$. The normalization condition $|\alpha|^2 + |\beta|^2 = 1$ is

$$\begin{aligned}1 &= |\alpha|^2 + |\beta|^2 = \bar{\alpha}\alpha + \bar{\beta}\beta \\ &= \langle\psi|0\rangle\langle 0|\psi\rangle + \langle\psi|1\rangle\langle 1|\psi\rangle \\ &= \langle\psi|\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)|\psi\rangle \\ &= \langle\psi|\psi\rangle \quad .\end{aligned} \tag{5}$$

The last equality above follows since $|0\rangle\langle 0| = \left(\begin{smallmatrix}1 & 0 \\ 0 & 0\end{smallmatrix}\right)$, $|1\rangle\langle 1| = \left(\begin{smallmatrix}0 & 0 \\ 0 & 1\end{smallmatrix}\right)$, so $|0\rangle\langle 0| + |1\rangle\langle 1|$ is the 2×2 identity matrix. (This trick is important enough to have its own name, the "resolution of the identity.")

3 Measurement in an arbitrary basis

We may choose any orthogonal basis $v, v^{\perp}$ and measure the qubit in it. To do this, we rewrite our state in that basis: $|\psi\rangle = \alpha'|v\rangle + \beta'|v^{\perp}\rangle$. The outcome is v with probability $|\alpha'|^2$, and $|v^{\perp}\rangle$ with probability $|\beta'|^2$. If the outcome of the measurement on $|\psi\rangle$ yields $|v\rangle$, then as before, the qubit is then in state $|v\rangle$.

We now illustrate this explicitly, using the new notations developed above. We measure $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in the $|v\rangle, |v^{\perp}\rangle$ basis, where $|v\rangle = a|0\rangle + b|1\rangle$. What is the probability of measuring $|v\rangle$?

Our measurement basis vector is $|v\rangle = a|0\rangle + b|1\rangle$. Let's choose the orthogonal complement as $|v^{\perp}\rangle = \bar{b}|0\rangle - \bar{a}|1\rangle$. Check that $\langle v|v^{\perp}\rangle = ab - ba = 0$.

In new basis, resolution of the identity can be written as

$$I = |v\rangle\langle v| + |v^\perp\rangle\langle v^\perp| \tag{6}$$

and

$$\begin{aligned}
|\psi\rangle &= \left(|v\rangle\langle v| + |v^\perp\rangle\langle v^\perp|\right)|\psi\rangle \\
&= \alpha(|v\rangle\langle v|0\rangle + |v^\perp\rangle\langle v^\perp|0\rangle) + \beta(|v\rangle\langle v|1\rangle + |v^\perp\rangle\langle v^\perp|1\rangle) \\
&= (\alpha\langle v|0\rangle + \beta\langle v|1\rangle)|v\rangle + (\alpha\langle v^\perp|0\rangle + \beta\langle v^\perp|1\rangle)|v^\perp\rangle \\
&= (\alpha\bar{a} + \beta\bar{b})|v\rangle + (\alpha b - \beta a)|v^\perp\rangle \ .
\end{aligned}$$

The probability of measuring $|v\rangle$ in a measurement in the $v, v^\perp$ basis is therefore

$$|\langle v|\psi\rangle|^2 = |\alpha\bar{a} + \beta\bar{b}|^2 \ .$$

4 Projections

We can choose basis vectors **i** and **j** in 2D

$$|\mathbf{i}\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \text{ and } |\mathbf{j}\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right). \tag{7}$$

Projection of ket onto basis vector given by inner product is

$$\langle i||v\rangle = \left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right) = a. \tag{8}$$

We can also write

$$|v\rangle = \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right) = \langle i||v\rangle|i\rangle + \langle j||v\rangle|j\rangle = |i\rangle\langle i||v\rangle + |j\rangle\langle j||v\rangle = I|v\rangle. \tag{9}$$

5 What is a bra?

Bra of $|v\rangle$ is NOT a vector in some complex vector space. Bra is defined by inner product $\langle a||b\rangle$ wich maps the vector $\langle v|$ onto the set of complex numbers

$$L_a(|b\rangle) = \langle a||b\rangle \in \mathscr{C}, \tag{10}$$

i.e.

$$\langle a| \equiv L_a \tag{11}$$

is a map from V to $\mathscr{C}$. Properties: (1) linearity: if $|c\rangle = |a\rangle + |b\rangle$, then $\langle c| = \langle a| + \langle b|$. (2) multiplication by a constant $c|c\rangle \mapsto \langle a|c^*$.

Usually we normalize kets. For

$$|tketv = \frac{v_x}{\sqrt{v_x^2 + v_y^2}}|0\rangle + \frac{v_y}{\sqrt{v_x^2 + v_y^2}}|1\rangle, \tag{12}$$

we work with vector

$$|tket\tilde{v} = v_x|0\rangle + v_y|1\rangle, \tag{13}$$

that has property $\langle\tilde{v}||\tilde{v}\rangle = 1$.

A single measurement of th observable A on a state $|\Psi\rangle$ in the basis (representation) of eigenstates of $\hat{A}$ will yield the value a_i, with probability $P_\Psi(a_i) = |\langle a_i|\Psi\rangle|^2$. This defines the measurement operator

$$\hat{M}_i = |a_i\rangle\langle a_i|$$

that acts on the state $|\Psi\rangle$. The normalized state after measurement is then easily seen to be equal to

$$\frac{\hat{M}_i|\Psi\rangle}{\sqrt{\langle\Psi|M_i^\dagger M_i|\Psi\rangle}}.$$

For a measurement in the $|a_i\rangle$ basis this is given by

$$|i\rangle\frac{\langle i||\Psi\rangle}{\sqrt{\langle\Psi|M_i^\dagger M_i|\Psi\rangle}},$$

where we have abbreviated

$$|a_i\rangle \equiv |i\rangle.$$

For example, suppose we have the linear superposition

$$|\Psi\rangle = \alpha_1|1\rangle + \alpha_2|2\rangle + \alpha_3|3\rangle + ... + \alpha_k|k\rangle.$$

Making a single measurement of the observable A on $|\Psi\rangle$ will result in the outcome a_i with probability

$$P_\Psi(a_i) = |\alpha_i|^2$$

and the resulting state after the measurement is equal to

$$|i\rangle\left(\frac{\alpha_i}{|\alpha_i|}\right).$$

The measurement of the observable has "collapsed" the state $|\Psi\rangle$ to a single eigenstate $|i\rangle \equiv |a_i\rangle$ of $\hat{A}$ (recall these constitute an orthonormal basis).