

Question 1

Sometimes we can prune down a network before inference to simplify our computations. In this problem, you will develop a condition under which "dangling" variables can be pruned.

- a. Consider two networks, G and G' . In G , we have nodes X, Y , and Z where X is the parent of Y and Y is the parent of Z , so that $P(x, y, z) = P(x) P(y | x) P(z | y)$. G' is identical to G except, Z and its CPTs have been deleted. Show that $P(x | y)$ is the same whether we compute it from G or G' .

- b. More generally, assume we have a network G in which there are nodes $X_1 \dots X_n$. Assume that we wish to calculate $P(Q_1 = q_1, \dots, Q_k = q_k | E_1 = e_1, \dots, E_m = e_m)$, where each query variable Q and evidence variable E is one of the variables X_i . Let H_1, \dots, H_p be all the remaining (hidden) nodes which are neither observed nor queried. Write an expression for $P(q_1, \dots, q_k | e_1, \dots, e_m)$ in terms of the full joint distribution entries $P(q_1, \dots, q_k, e_1, \dots, e_m, h_1, \dots, h_p)$.

- c. Assume that one of the hidden variables H , say H_1 , is a leaf node, that is, it has no children in G . Let G' be identical to G but with H removed. Show that $P(q_1, \dots, q_k | e_1, \dots, e_m)$ is the same whether calculated in G or G' .

- d. Prove that any node which does not dominate either an evidence or query node ("dangling nodes") may be pruned without effecting the result of a query.

Question 2 – HMM and Particle Filtering

You sometimes get colds, which make you sneeze. You also get allergies, which make you sneeze. Sometimes you are well, which doesn't make you sneeze. You decide to model the process using the following HMM, with hidden states $X \in \{well, allergy, cold\}$ and observations $E \in \{well, allergy, cold\}$:

$$P(X_1)$$

<i>well</i>	1
<i>allergy</i>	0
<i>cold</i>	0

$$P(X_t | X_{t-1} = well)$$

<i>well</i>	0.7
<i>allergy</i>	0.2
<i>cold</i>	0.1

$$P(X_t | X_{t-1} = allergy)$$

<i>well</i>	0.6
<i>allergy</i>	0.3
<i>cold</i>	0.1

$$P(X_t | X_{t-1} = cold)$$

<i>well</i>	0.2
<i>allergy</i>	0.2
<i>cold</i>	0.6

Transitions

$$P(E_t | X_t = well)$$

<i>quiet</i>	1.0
<i>sneeze</i>	0.0

$$P(E_t | X_t = allergy)$$

<i>quiet</i>	0.0
<i>sneeze</i>	1.0

$$P(E_t | X_t = cold)$$

<i>quiet</i>	0.0
<i>sneeze</i>	1.0

Emissions

Note that colds are "stickier" in that you tend to have them for multiple days, while allergies come and go on a quicker time scale. However, allergies are more frequent. Assume that on the first day, you are well.

a) Imagine that you observe the sequence quiet, sneeze, sneeze. What is the probability that you were well all three days and observed these effects?

b) What is the posterior distribution over your state on day 2 (X_2) if $E_1 = quiet$, $E_2 = sneeze$?

c) What is the posterior distribution over your state on day 3 (X_3) if $E_1 = \text{quiet}$, $E_2 = \text{sneeze}$, $E_3 = \text{sneeze}$?

d) What is the Viterbi (most likely) sequence for the observation sequence quiet, sneeze, sneeze, sneeze, quiet, quiet, sneeze, quiet, quiet? [Hint: you should not have to do extensive calculations.]

Imagine you are monitoring your state using the particle filtering algorithm, and on a given day you have 5 particles on *well*, 2 on *cold*, and 3 on *allergy* before making an observation on that day.

e) If you observe sneeze, what weight will each of your particles have?

f) After resampling, what is the expected number of particles you will have on cold?