CS 188 Homework 1 - Solutions for 13.8, 14.3, 14.4

Problem 13.8

Let D and T be random variables for disease and test, respectively.

$$\begin{aligned}
P(D = true|T = true) & \\
&= \frac{P(T{=}true|D{=}true)P(D{=}true)}{P(T{=}true)} \\
&= \frac{P(T{=}true|D{=}true)P(D{=}true)}{P(T{=}true|D{=}true)P(D{=}true)+P(T{=}true|D{=}false)P(D{=}false)} \\
&= \frac{0.99 \cdot 0.0001}{0.99 \cdot 0.0001 + 0.01 \cdot 0.9999} \\
&= \frac{1}{102}
\end{aligned}$$

Since the disease is rare, the probability you have it is still small.

Problem 14.3

(a)

(i) Incorrect. Consider conditioning on M_1 and M_2. We have that N is conditionally independent of F_1 and F_2 given M_1 and M_2. To see this, note that N is not reachable from F_1 or F_2 when M_1 and M_2 are shaded (the Bayes' ball algorithm). Hence, in this network $P(N|M_1, M_2, F_1, F_2) = P(N|M_1, M_2)$. But this is false. For example, $P(N = 1|M_1 = 1, M_2 = 1, F_1 = true, F_2 = true) = 0$ as both measurements have undercounted. But $P(N = 1|M_1 = 1, M_2 = 1)$ is nonzero.

By the chain rule, we can factor any probability distribution $P(A, B, C, D, ..., Z)$ as

$$P(A, B, C, D, ..., Z) = P(A|B, C, D, ..., Z)P(B|C, D, ...Z,)P(C|D, ..., Z) \cdot ... \cdot P(Z)$$

(ii) Correct. Consider the ordering of nodes M_1, M_2, F_1, F_2, N.

$$\begin{aligned}
P(M_1, M_2, F_1, F_2, N) = \; & P(M_1|M_2, F_1, F_2, N)P(M_2|F_1, F_2, N)P(F_1|F_2, N)P(F_2|N)P(N) \\
= \; & P(M_1|F_1, N)P(M_2|F_2, N)P(F_1)P(F_2)P(N)
\end{aligned}$$

since M_1 depends only on F_1 and N (and likewise for M_2), and F_1, F_2 are independent of N. This last line is the probability distribution represented by the Bayes' net.

(iii) Correct. Consider the ordering of nodes F_2, F_1, N, M_2, M_1

$$\begin{aligned} P(F_2, F_1, N, M_2, M_1) = & \; P(F_2|F_1, N, M_2, M_1)P(F_1|N, M_2, M_1)P(N|M_2, M_1)P(M_2|M_1)P(M_1) \\ = & \; P(F_2|N, M_2)P(F_1|N, M_1)P(N|M_2, M_1)P(M_2|M_1)P(M_1) \end{aligned}$$

This last line is the probability distribution represented by the Bayes' net.

(b) Network (ii) is the best since it is the sparsest (most efficient) correct network.

(c)

N	M_1	$P(M_1\|N)$
1	0	$e + f - ef$
1	1	$(1-2e)(1-f)$
1	2	$e(1-f)$
1	3	0
1	4	0
2	0	f
2	1	$e(1-f)$
2	2	$(1-2e)(1-f)$
2	3	$e(1-f)$
2	4	0
3	0	f
3	1	0
3	2	$e(1-f)$
3	3	$(1-2e)(1-f)$
3	4	$e(1-f)$

Note that the meaning of e is slightly ambiguous in the problem. It could be that there is a probility e of overcounting by one and a probability e of undercounting by one. Or it could be the probability of each is $e/2$. Under this interpretation simply replace e by $e/2$ in the above table.

(d) $M_1 = 1$ and $M_2 = 3$. $N >= 0$ as we cannot have a negative number of stars.
Can N be $<= 1$? No, M_2 cannot overcount by 2 or more.
Can $N = 2$? Yes, if both M_1 and M_2 are off by one (errors occur).

Can $N = 3$? No, M_1 cannot undercount by exactly 2 (only 1 or more than 2).
Can $N = 4$? Yes, if $F_1 = true$, and M_2 makes an error.
Can $N = 5$? No, M_2 cannot undercount by exactly 2.
Can N be $>= 6$? Yes, if both $F_1 = F_2 = true$.
So $N = 2$, or $N = 4$, or $N >= 6$.

(e) $P(N)$, the prior on the number of stars, is the additional information required to compute the most likely number.

Problem 14.4

$$P(N = n|M_1 = 2, M_2 = 2) \propto P(N = n, M_1 = 2, M_2 = 2)$$

$P(N = n, M_1 = 2, M_2 = 2)$
$= \sum_{f1,f2} P(N = n, M_1 = 2, M_2 = 2, F_1 = f_1, F_2 = f_2)$
$= \sum_{f1,f2} P(M_1 = 2|N = n, F_1 = f_1)P(M_2 = 2|N = n, F_2 = f_2)P(F_1 = f_1)P(F_2 = f_2)P(N = n)$
$= \sum_{f1} \sum_{f2} P(M_1 = 2|N = n, F_1 = f_1)P(M_2 = 2|N = n, F_2 = f_2)P(F_1 = f_1)P(F_2 = f_2)P(N = n)$
$= \sum_{f1} \sum_{f2} P(N = n)P(M_1 = 2|N = n, F_1 = f_1)P(F_1 = f_1)P(M_2 = 2|N = n, F_2 = f_2)P(F_2 = f_2)$
$= P(N = n) \sum_{f1} P(M_1 = 2|N = n, F_1 = f_1)P(F_1 = f_1) \sum_{f2} P(M_2 = 2|N = n, F_2 = f_2)P(F_2 = f_2)$
$= P(N = n) \sum_{f1} P(M_1 = 2|N = n, F_1 = f_1)P(F_1 = f_1)P(M_2 = 2|N = n)$
$= P(N = n)P(M_1 = 2|N = n)P(M_2 = 2|N = n)$

Using the table from part (c), we have:
$P(N = 1, M_1 = 2, M_2 = 2) = [e(1 - f)]^2 \cdot P(N = 1)$
$P(N = 2, M_1 = 2, M_2 = 2) = [(1 - 2e)(1 - f)]^2 \cdot P(N = 2)$
$P(N = 3, M_1 = 2, M_2 = 2) = [e(1 - f)]^2 \cdot P(N = 3)$

Defining $Z = P(M_1 = 2, M_2 = 2) = \sum_{n\in\{1,2,3\}} P(N = n, M_1 = 2, M_2 = 2)$ we have:

$P(N = n|M_1 = 2, M_2 = 2) = P(N = n, M_1 = 2, M_2 = 2)/Z$ for $n \in \{1, 2, 3\}$.

Note that since the question did not specify $P(N)$, it is ok if you assumed a uniform prior ($P(N = 1) = P(N = 2) = P(N = 3) = \frac{1}{3}$).