CS 188 Section Handout: Probability Basics

Let $D \in \{d_0, d_1, d_2, d_3\}$ be one of four dice, where d_0 is fair. Let $R \in [1, 6]$ be the outcome of a die roll.

1 Joint probability assembly

A casino employee informs you (the inspector) about a dealer, Angelo: "He cheats a third of the time, using loaded die d_1 for which the six-dotted side shows up five times as often as each of the other sides."

D	$P(D)$
d_0	2/3
d_1	1/3

R	D	$P(R\|D)$
1	d_0	1/6
2	d_0	1/6
3	d_0	1/6
4	d_0	1/6
5	d_0	1/6
6	d_0	1/6
1	d_1	1/10
2	d_1	1/10
3	d_1	1/10
4	d_1	1/10
5	d_1	1/10
6	d_1	1/2

R	D	$P(R, D)$
1	d_0	1/9
2	d_0	1/9
3	d_0	1/9
4	d_0	1/9
5	d_0	1/9
6	d_0	1/9
1	d_1	1/30
2	d_1	1/30
3	d_1	1/30
4	d_1	1/30
5	d_1	1/30
6	d_1	1/6

2 Estimation

Your informant reports back with the following statistics about another dealer, Bert: "He uses two dice, both loaded. Using d_2, I watced him roll 15 sixes, 10 fives, 5 fours, 5 threes, 5 twos, and no ones. With d_3, I observed 5 sixes, 10 fives, 15 fours, 15 threes, 15 twos, and 20 ones."

R	D	$c(R, D)$
1	d_0	0
2	d_0	5
3	d_0	5
4	d_0	5
5	d_0	10
6	d_0	15
1	d_1	20
2	d_1	15
3	d_1	15
4	d_1	15
5	d_1	10
6	d_1	5

R	D	$P(R, D)$
1	d_0	0
2	d_0	1/24
3	d_0	1/24
4	d_0	1/24
5	d_0	1/12
6	d_0	1/8
1	d_1	1/6
2	d_1	1/8
3	d_1	1/8
4	d_1	1/8
5	d_1	1/12
6	d_1	1/24

3 Inference by enumeration

- What is $P(R = 6 \mid \text{dealer} = A)$? Sum over d_0 and d_1: $\frac{1}{9} + \frac{1}{6} = \frac{5}{18}$
- What is $P(R = 6 \mid \text{dealer} = B)$? Sum over d_2 and d_3: $\frac{1}{8} + \frac{1}{24} = \frac{1}{6}$
- What is $P(D = d_0 \mid R = 6, \text{dealer} = A)$? $\frac{P(D=d_0,\ R=6 \mid \text{dealer}=A)}{P(R=6 \mid \text{dealer}=A)} = \frac{1/9}{5/18} = \frac{2}{5}$
- What is $P(D = d_0 \mid \text{dealer} = B)$? 0
- What is $P(D = d_2 \mid \text{dealer} = B)$? $\frac{40}{120} = \frac{1}{3}$.

Suppose you know that A works some five nights per week and B works on some other two nights.

- What is $P(D)$?

  As an example, consider $P(D = d_0)$.

$$\begin{aligned} P(D = d_0) &= P(D = d_0,\ dealer = A) + P(D = d_0,\ dealer = B) \\ &= P(D = d_0 \mid dealer = A)P(dealer = A) + P(D = d_0 \mid dealer = B)P(dealer = B) \\ &= \frac{2}{3} \cdot \frac{5}{7} + 0 \cdot \frac{2}{7} \\ &= \frac{10}{21} \end{aligned}$$

D	$P(D)$
d_0	10/21
d_1	5/21
d_2	2/21
d_3	4/21

- One night, you observe a dealer (A or B) roll a six. What is the probability that the die is loaded?

$$\begin{aligned} P(D \in \{d_1, d_2, d_3\} | R = 6) &= 1 - P(D = d_0 | R = 6) \\ &= 1 - [P(D = d_0 | R = 6, A)P(A) + P(D = d_0 | R = 6, B)P(B)] \\ &= 1 - [\tfrac{2}{5} \cdot \tfrac{2}{7} + 0 \cdot \tfrac{2}{7}] = \tfrac{5}{7} \end{aligned}$$

4 Sequence of independent events

You confront the suspected dealer and ask him to roll his current die three times in a row. He rolls the sequence $S = (6, 2, 6)$

Calculate the following likelihood probabilities:

- $P(S \mid D = d_0) = (\frac{1}{6})^3 = \frac{1}{216}$
- $P(S \mid D = d_1) = (\frac{1}{2}) \cdot (\frac{1}{2}) \cdot (\frac{1}{10}) = \frac{1}{40}$
- $P(S \mid D = d_2) = (\frac{1}{8}) \cdot (\frac{3}{8}) \cdot (\frac{3}{8}) = \frac{9}{512}$
- $P(S \mid D = d_3) = (\frac{1}{16})^3 = \frac{1}{4096}$

5 Bayes' Rule

Given the evidence you acquired previously, determine which die was rolled:

$$\hat{d} = \arg\max_{d} P(D = d \mid S)$$

Is the die loaded? Which dealer are you arresting?

By Bayes rule, we have

$$P(D = d \mid S = s) = \frac{P(S = s \mid D = d)P(D = d)}{P(S = s)} \tag{1}$$

Note that for any d, the denominator of the right hand side will be the same. So, when we look for the maximal d, we need only compute the numerator for each die.

- $P(S \mid D = d_0)P(D = d_0) = \frac{1}{216} \cdot \frac{10}{21} = \frac{5}{2268}$
- $P(S \mid D = d_1)P(D = d_1) = \frac{1}{40} \cdot \frac{5}{21} = \frac{1}{168}$
- $P(S \mid D = d_2)P(D = d_2) = \frac{9}{512} \cdot \frac{2}{21} = \frac{3}{1792}$
- $P(S \mid D = d_3)P(D = d_3) = \frac{1}{4096} \cdot \frac{4}{21} = \frac{1}{21504}$

The greatest of these values arises from $D = d_1$, so the most likely scenario is that you are confronting A using the loaded d_1. Time to call in some backup!

Note: The denominator can be expressed as the sum of numerators:

$$P(S) = \sum_{d} P(S|D = d)P(D = d) = \frac{5}{2268} + \frac{1}{168} + \frac{3}{1792} + \frac{1}{21504}$$

Then use Bayes' Rule to determine that there is roughly a 60% chance that you are correctly accusing dealer A. On the other hand, there is a 22% chance that dealer A was rolling the fair die, and an 18% chance that it's really dealer B. Maybe you should gather more evidence...