CS 188: Artificial Intelligence
Fall 2007

Lecture 10: MDPs

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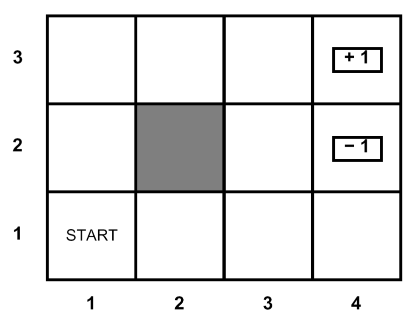
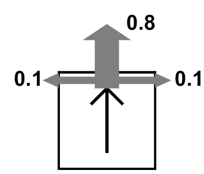
Markov Decision Processes

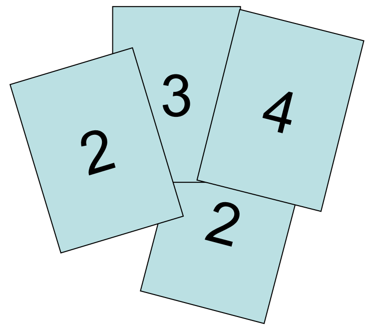
- An MDP is defined by:
  - A set of states $s \in S$
  - A set of actions $a \in A$
  - A transition function T(s,a,s')
    - Prob that a from s leads to s'
    - i.e., P(s' | s,a)
    - Also called the model
  - A reward function R(s, a, s')
    - Sometimes just R(s) or R(s')
  - A start state (or distribution)
  - Maybe a terminal state
- MDPs are a family of non-deterministic search problems
  - Reinforcement learning: MDPs where we don't know the transition or reward functions





Example: High-Low

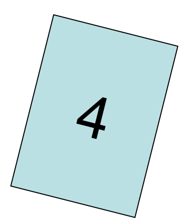
- Three card types: 2, 3, 4
- Infinite deck, twice as many 2's
- Start with 3 showing
- After each card, you say "high" or "low"
- New card is flipped
- If you're right, you win the points shown on the new card
- Ties are no-ops
- If you're wrong, game ends



- Differences from expectimax:
  - #1: get rewards as you go
  - #2: you might play forever!

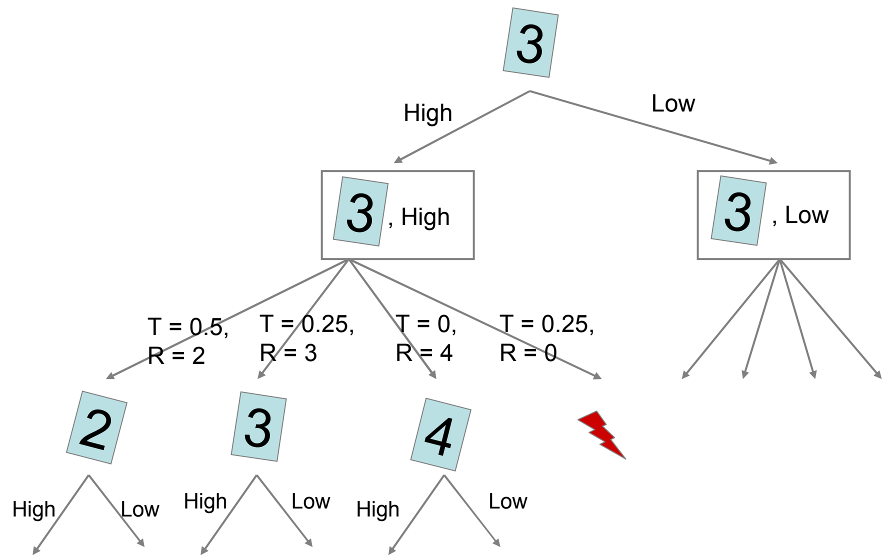
High-Low

- States: 2, 3, 4, done
- Actions: High, Low
- Model: T(s, a, s'):
  - P(s'=done | 4, High) = 3/4
  - P(s'=2 | 4, High) = 0
  - P(s'=3 | 4, High) = 0
  - P(s'=4 | 4, High) = 1/4
  - P(s'=done | 4, Low) = 0
  - P(s'=2 | 4, Low) = 1/2
  - P(s'=3 | 4, Low) = 1/4
  - P(s'=4 | 4, Low) = 1/4
  - …
- Rewards: R(s, a, s'):
  - Number shown on s' if $s \neq s'$
  - 0 otherwise
- Start: 3



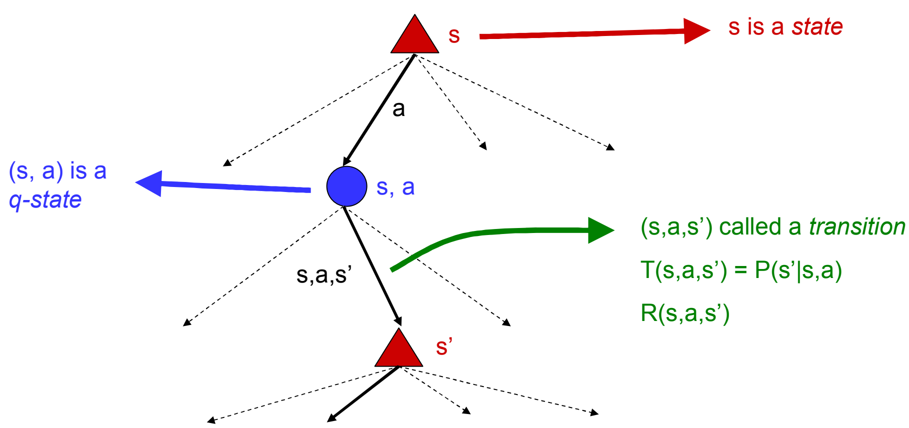
Note: could choose actions with search. How?

Example: High-Low



MDP Search Trees

- Each MDP state gives an expectimax-like search tree



Utilities of Sequences

- In order to formalize optimality of a policy, need to understand utilities of sequences of rewards
- Typically consider stationary preferences:

$$[r, r_0, r_1, r_2, \ldots] \succ [r, r'_0, r'_1, r'_2, \ldots]$$
$$\Leftrightarrow$$
$$[r_0, r_1, r_2, \ldots] \succ [r'_0, r'_1, r'_2, \ldots]$$

Assuming that reward depends only on state for these slides!

- Theorem: only two ways to define stationary utilities
  - Additive utility:

$$V([s_0, s_1, s_2, \ldots]) = R(s_0) + R(s_1) + R(s_2) + \cdots$$

  - Discounted utility:

$$V([s_0, s_1, s_2, \ldots]) = R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) \cdots$$

Infinite Utilities?!

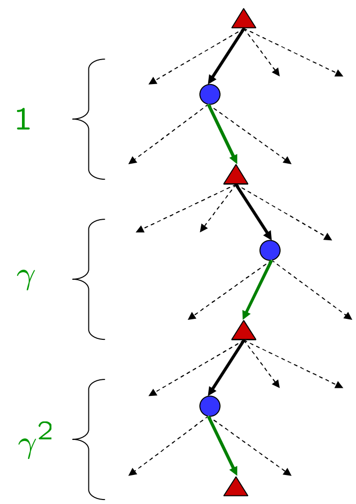
- Problem: infinite sequences with infinite rewards
- Solutions:
  - Finite horizon:
    - Terminate after a fixed T steps
    - Gives nonstationary policy (π depends on time left)
  - Absorbing state(s): guarantee that for every policy, agent will eventually "die" (like "done" for High-Low)
  - Discounting: for $0 < \gamma < 1$

$$V([s_0, \ldots s_\infty]) = \sum_{t=0}^{\infty} \gamma^t R(s_t) \leq R_{\text{max}}/(1-\gamma)$$

    - Smaller γ means smaller "horizon" – shorter term focus

Discounting

- Typically discount rewards by $\gamma < 1$ each time step
  - Sooner rewards have higher utility than later rewards
  - Also helps the algorithms converge



Episodes and Returns

- An epsiode is a run of an MDP
  - Sequence of transitions (s,a,s')
  - Starts at start state
  - Ends at terminal state (if it ends)
  - Stochastic!

- The utility, or return, of an epsiode
  - The discounted sum of the rewards

$$\sum_i \gamma^i R(s_i, a_i, s_{i+1})$$

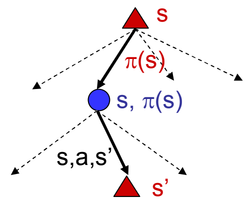
Utilities under Policies

- Fundamental operation: compute the utility of a state s

- Define the value (utility) of a state s, under a fixed policy π:

  $V^\pi(s)$ = expected return starting in s and following π

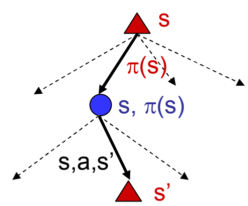
- Recursive relation (one-step look-ahead):



$$V^\pi(s) = \sum_{s'} T(s,\pi(s),s')[R(s,\pi(s),s') + \gamma V^\pi(s')]$$

Policy Evaluation

- How do we calculate values for a fixed policy?
- Idea one: it's just a linear system, solve with Matlab (or whatever)
- Idea two: turn recursive equations into updates
  - $V_i^\pi(s)$ = expected returns over the next i transitions while following π



$$V_0^\pi(s) = 0$$

$$V_{i+1}^\pi(s) \leftarrow \sum_{s'} T(s,\pi(s),s')[R(s,\pi(s),s') + \gamma V_i^\pi(s')]$$

Equivalent to doing depth i search and plugging in zero at leaves

Example: High-Low

- Policy: always say "high"
- Iterative updates:

$$V_0 = \{2:0, \quad 3:0, \quad 4:0, \quad d:0\}$$

$$V_1(2) = \frac{1}{2}(R(2,H,2) + V_0(2)) + \frac{1}{4}(R(2,H,3) + V_0(3)) +$$

$$\frac{1}{4}(R(2,H,4) + V_0(4)) + \ 0(R(2,H,d) + V_0(d))$$

$$V_1(2) = \frac{1}{2}(0+0) + \frac{1}{4}(3+0) + \frac{1}{4}(4+0) + 0(0+0)$$

$$V_1(2) = \frac{7}{4}$$

$$V_1 = \{2:\frac{7}{4}, \quad 3:1, \quad 4:0, \quad d:0\}$$

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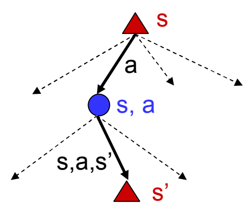
Q-Functions

- Also, define a q-value, for a state and action (q-state)
  - $Q^\pi(s)$ = expected return starting in s, taking action a and following π thereafter



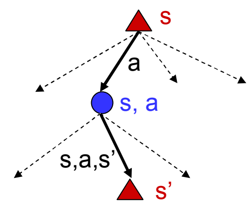
$$Q^\pi(s,a) = \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^\pi(s')]$$

$$V^\pi(s) = Q^\pi(s,\pi(s))$$

$$Q^\pi(s,a) = \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma Q^\pi(s',\pi(s'))]$$

Recap: MDP Quantities

- Return = Sum of future discounted rewards in one episode (stochastic)



- V: Expected return from a state under a policy

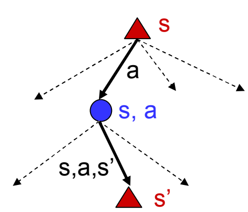
$$V^{\pi}(s) = Q^{\pi}(s, \pi(s))$$

- Q: Expected return from a q-state under a policy

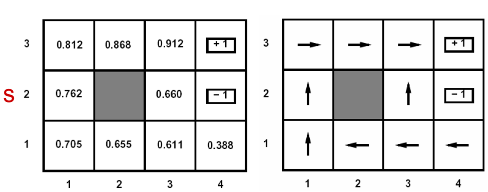
$$Q^{\pi}(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^{\pi}(s') \right]$$

Optimal Utilities

- Fundamental operation: compute the optimal utilities of states s



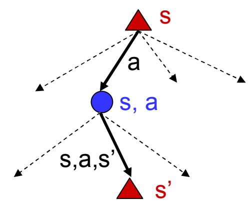
- Define the utility of a state s:
  $V^*(s)$ = expected return starting in s and acting optimally
- Define the utility of a q-state (s,a):
  $Q^*(s)$ = expected return starting in s, taking action a and thereafter acting optimally
- Define the optimal policy:
  $\pi^*(s)$ = optimal action from state s



The Bellman Equations

- Definition of utility leads to a simple relationship amongst optimal utility values:

  Optimal rewards = maximize over first action and then follow optimal policy



- Formally:

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

Solving MDPs

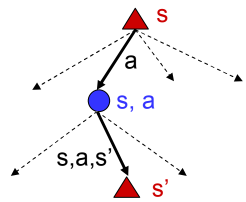
- We want to find the optimal policy π

- Proposal 1: modified expectimax search:

$$\pi(s) = \arg\max_a Q^*(s,a)$$

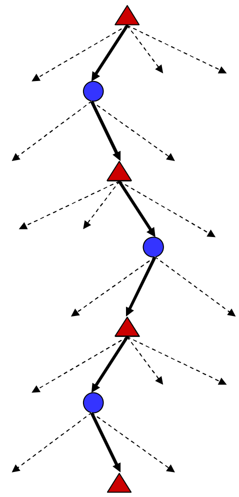
$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

$$V^*(s) = \max_a Q^*(s,a)$$



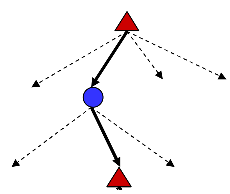
MDP Search Trees?

- Problems:
  - This tree is usually infinite (why?)
  - The same states appear over and over (why?)
  - There's actually one tree per state (why?)
- Ideas:
  - Compute to a finite depth (like expectimax)
  - Consider returns from sequences of increasing length
  - Cache values so we don't repeat work



Value Estimates

- Calculate estimates $V_k^*(s)$
  - Not the optimal value of s!
  - The optimal value considering only next k time steps (k rewards)
  - As $k \rightarrow \infty$, it approaches the optimal value
  - Why:
    - If discounting, distant rewards become negligible
    - If terminal states reachable from everywhere, fraction of episodes not ending becomes negligible
    - Otherwise, can get infinite expected utility and then this approach actually won't work



Memoized Recursion?

- Recurrences:

$$V_0^*(s) = 0$$

$$V_i^*(s) = \max_a Q_i^*(s,a)$$

$$Q_i^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_{i-1}^*(s')\right]$$

$$\pi_i(s) = \arg\max_a Q_i^*(s,a)$$

- Cache all function call results so you never repeat work
- What happened to the evaluation function?

Value Iteration

- Problems with the recursive computation:
  - Have to keep all the $V_k^*(s)$ around all the time
  - Don't know which depth $\pi_k(s)$ to ask for when planning
- Solution: value iteration
  - Calculate values for all states, bottom-up
  - Keep increasing k until convergence

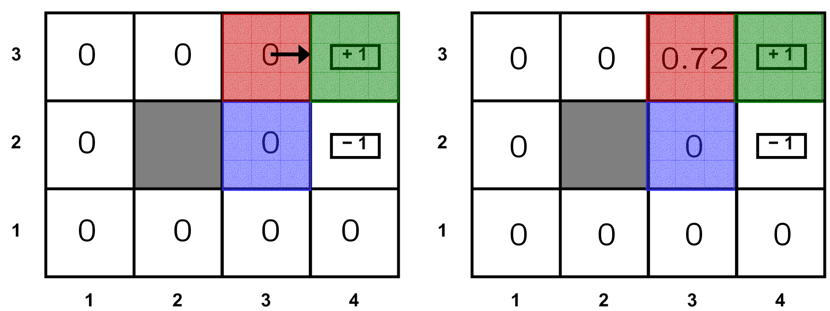
Value Iteration

- Idea:
  - Start with $V_0^*(s) = 0$, which we know is right (why?)
  - Given V_i^*, calculate the values for all states for depth i+1:

$$V_{i+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_i(s')\right]$$

  - This is called a value update or Bellman update
  - Repeat until convergence
- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - Policy may converge long before values do

Example: Bellman Updates



$$V_{i+1}(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_i(s')\right]$$

$$V_{i+1}(\langle 3,3\rangle) = \sum_{s'} T(\langle 3,3\rangle, \text{right}, s') \left[R(\langle 3,3\rangle) + 0.9\, V_i(s')\right]$$

$$= 0.9\,[0.8 \cdot 1 + 0.1 \cdot 0 + 0.1 \cdot 0]$$

Example: Value Iteration

V_2

	1	2	3	4
3	0	0	0.72	+1
2	0		0	−1
1	0	0	0	0

V_3

	1	2	3	4
3	0	0.52	0.78	+1
2	0		0.43	−1
1	0	0	0	0

- Information propagates outward from terminal states and eventually all states have correct value estimates

[DEMO]

Convergence*

- Define the max-norm: $||U|| = \max_s |U(s)|$
- Theorem: For any two approximations U and V

$$||U^{t+1} - V^{t+1}|| \leq \gamma \, ||U^t - V^t||$$

  - I.e. any distinct approximations must get closer to each other, so, in particular, any approximation must get closer to the true U and value iteration converges to a unique, stable, optimal solution
- Theorem:

$$||U^{t+1} - U^t|| < \epsilon, \Rightarrow ||U^{t+1} - U|| < 2\epsilon\gamma/(1-\gamma)$$

  - I.e. once the change in our approximation is small, it must also be close to correct

Policy Iteration

- Alternative approach:
  - Step 1: Policy evaluation: calculate utilities for a fixed policy (not optimal utilities!) until convergence
  - Step 2: Policy improvement: update policy based on resulting converged (but not optimal!) utilities
  - Repeat steps until policy converges

- This is policy iteration
  - Can converge faster under some conditions

Policy Iteration

- Policy evaluation: with fixed current policy π, find values with simplified Bellman updates:
  - Iterate until values converge

$$V_{i+1}^{\pi_k}(s) \leftarrow \sum_{s'} T(s, \pi_k(s), s') \left[R(s, \pi_k(s), s') + \gamma V_i^{\pi_k}(s')\right]$$

- Policy improvement: with fixed utilities, find the best action according to one-step look-ahead

$$\pi_{k+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_k}(s')\right]$$

Comparison

- In value iteration:
  - Every pass (or "backup") updates both utilities (explicitly, based on current utilities) and policy (possibly implicitly, based on current policy)
- In policy iteration:
  - Several passes to update utilities with frozen policy
  - Occasional passes to update policies
- Hybrid approaches (asynchronous policy iteration):
  - Any sequences of partial updates to either policy entries or utilities will converge if every state is visited infinitely often