

CS-184: Computer Graphics

Lecture #3: 2D Transformations

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Today

- 2D Transformations
 - “Primitive” Operations
 - Scale, Rotate, Shear, Flip, Translate
 - Homogenous Coordinates
 - SVD
 - Start thinking about rotations...

Introduction

- **Transformation:**
An operation that changes one configuration into another
- **For images, shapes, etc.**
A geometric transformation maps positions that define the object to other positions

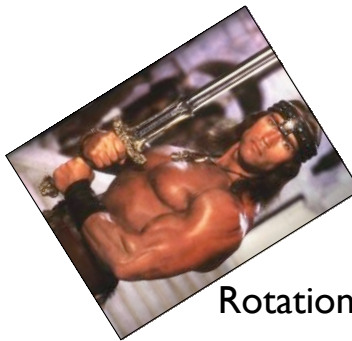
Linear transformation means the transformation is defined by a linear function... which is what matrices are good for.

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Some Examples



Original



Rotation



Uniform Scale



Nonuniform Scale

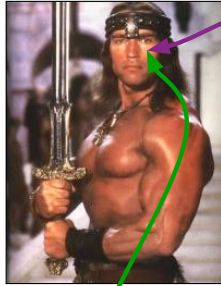


Shear

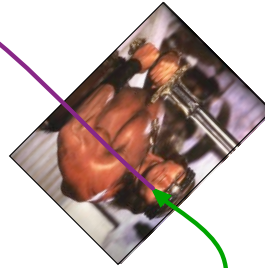
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Mapping Function

$f(x) = x$ in old image



$$c(x) = [195, 120, 58]$$



$$c'x = c(f(x))$$

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Linear -vs- Nonlinear



Nonlinear (swirl)



Linear (shear)

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Geometric -vs- Color Space



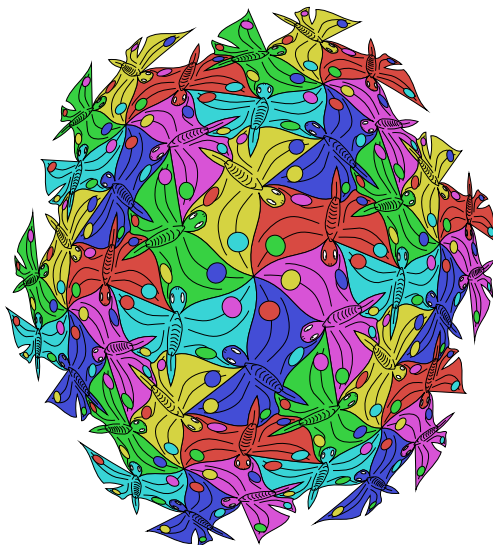
Color Space Transform
(edge finding)



Linear Geometric
(flip)

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Instancing

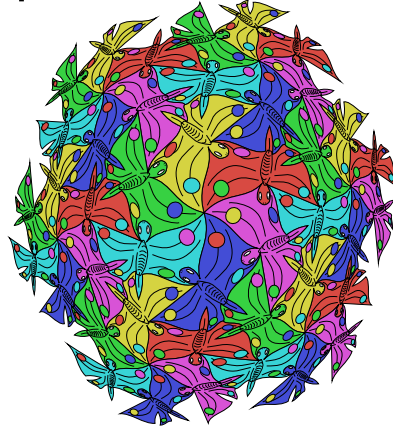


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Instancing

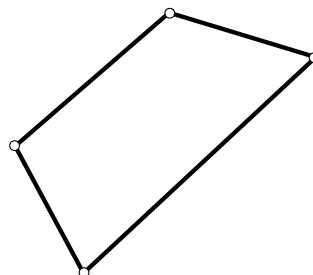
- Reuse geometric descriptions
- Saves memory



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Linear is Linear

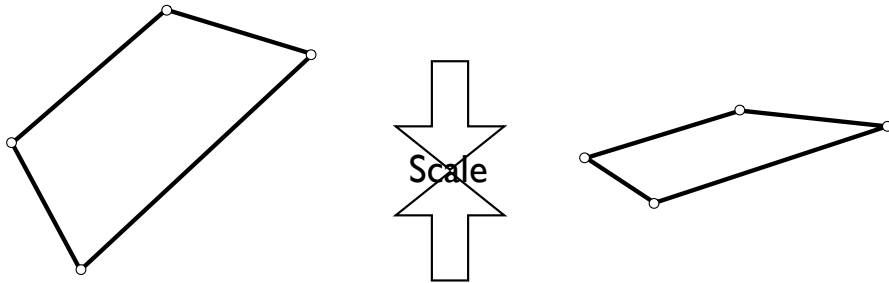
- Polygons defined by points
- Edges defined by interpolation between two points
- Interior defined by interpolation between all points
- *Linear* interpolation



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Linear is Linear

- Composing two linear function is still linear
- Transform polygon by transforming vertices



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Linear is Linear

- Composing two linear function is still linear
- Transform polygon by transforming vertices

$$f(x) = a + bx \quad g(f) = c + df$$

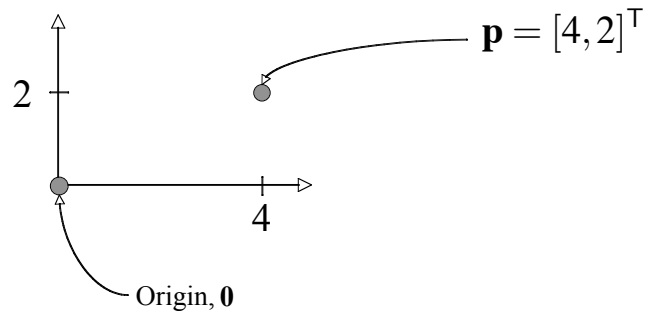
$$g(x) = c + df(x) = c + ad + bdx$$

$$g(x) = a' + b'x$$

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Points in Space

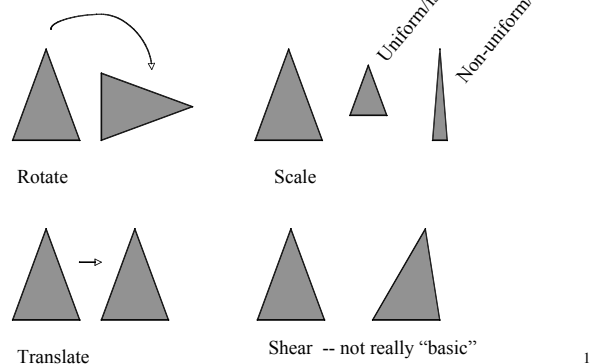
- Represent point in space by vector in R^n
 - Relative to some origin!
 - Relative to some coordinate axes!
- Later we'll add something extra...



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Basic Transformations

- Basic transforms are: rotate, scale, and translate
- Shear is a composite transformation!



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Linear Functions in 2D

$$x' = f(x, y) = c_1 + c_2x + c_3y$$

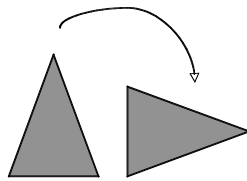
$$y' = f(x, y) = d_1 + d_2x + d_3y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} M_{xx} & M_{xy} \\ M_{yx} & M_{yy} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{t} + \mathbf{M} \cdot \mathbf{x}$$

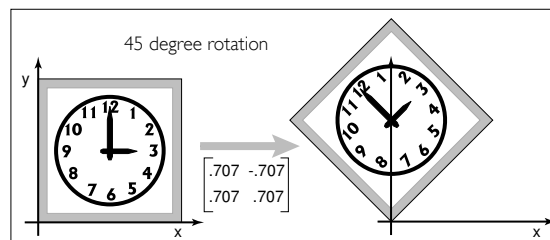
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Rotations



Rotate

$$\mathbf{p}' = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \mathbf{p}$$



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Rotations

- Rotations are positive counter-clockwise
- Consistent w/ right-hand rule
- Don't be different...
- Note:
 - rotate by zero degrees give identity
 - rotations are modulo 360 (or 2π)

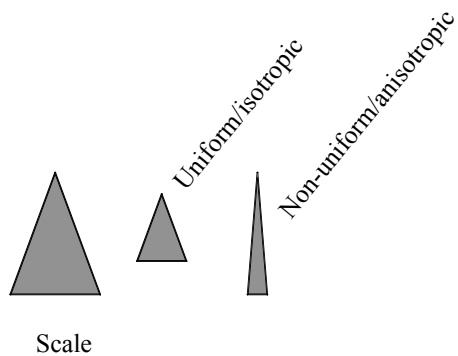
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Rotations

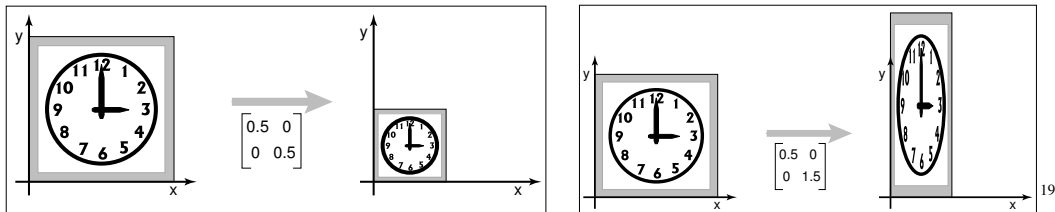
- Preserve lengths and distance to origin
- Rotation matrices are orthonormal
- $\text{Det}(\mathbf{R}) = 1 \neq -1$
- In 2D rotations commute...
 - But in 3D they won't!

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Scales

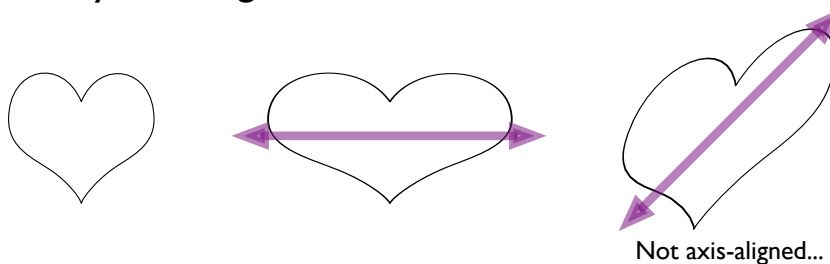


$$\mathbf{p}' = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \mathbf{p}$$

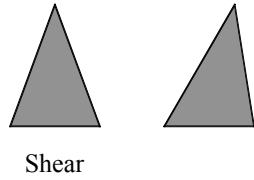


Scales

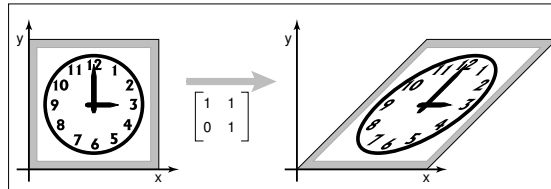
- Diagonal matrices
 - Diagonal parts are scale in X and scale in Y directions
 - Negative values flip
 - Two negatives make a positive (180 deg. rotation)
 - Really, axis-aligned scales



Shears



$$\mathbf{p}' = \begin{bmatrix} 1 & H_{yx} \\ H_{xy} & 1 \end{bmatrix} \mathbf{p}$$



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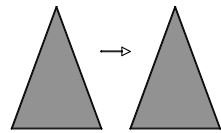
Shears

- Shears are not really primitive transforms
- Related to non-axis-aligned scales
- More shortly.....

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Translation

- This is the not-so-useful way:



Translate

$$\mathbf{p}' = \mathbf{p} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

Note that its not like the others.

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Arbitrary Matrices

- For everything but translations we have:

$$\mathbf{x}' = \mathbf{A} \cdot \mathbf{x}$$

- Soon, translations will be assimilated as well
- What does an arbitrary matrix mean?

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Singular Value Decomposition

- For any matrix, \mathbf{A} , we can write SVD:

$$\mathbf{A} = \mathbf{Q}\mathbf{S}\mathbf{R}^T$$

where \mathbf{Q} and \mathbf{R} are orthonormal and \mathbf{S} is diagonal

- Can also write Polar Decomposition

$$\mathbf{A} = \mathbf{Q}\mathbf{R}\mathbf{S}\mathbf{R}^T$$

where \mathbf{Q} is still orthonormal

not the same \mathbf{Q}

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Decomposing Matrices

- We can force \mathbf{Q} and \mathbf{R} to have $\text{Det}=1$ so they are rotations
- Any matrix is now:
 - Rotation:Rotation:Scale:Rotation
 - See, shear is just a mix of rotations and scales

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Composition

- Matrix multiplication composites matrices

$$\mathbf{p}' = \mathbf{B}\mathbf{A}\mathbf{p}$$

“Apply \mathbf{A} to \mathbf{p} and then apply \mathbf{B} to that result.”

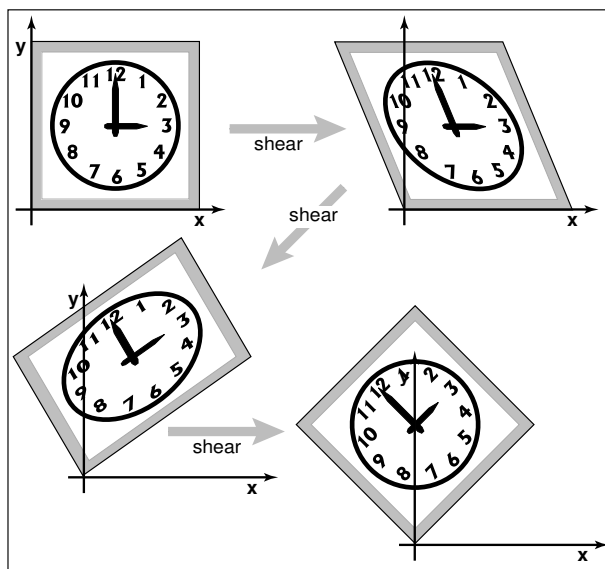
$$\mathbf{p}' = \mathbf{B}(\mathbf{A}\mathbf{p}) = (\mathbf{B}\mathbf{A})\mathbf{p} = \mathbf{C}\mathbf{p}$$

- Several translations composted to one
- Translations still left out...

$$\mathbf{p}' = \mathbf{B}(\mathbf{A}\mathbf{p} + \mathbf{t}) = \mathbf{A}\mathbf{p} + \mathbf{B}\mathbf{t} = \mathbf{C}\mathbf{p} + \mathbf{u}$$

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Composition



Transformations built up from others

SVD builds from scale and rotations

Also build other ways

i.e. 45 deg rotation built from shears

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Homogeneous Coordinates

- Move to one higher dimensional space
 - Append a 1 at the end of the vectors

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} \quad \tilde{\mathbf{p}} = \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$$

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Homogeneous Translation

$$\tilde{\mathbf{p}}' = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$$

$$\tilde{\mathbf{p}}' = \tilde{\mathbf{A}} \tilde{\mathbf{p}}$$

The tildes are for clarity to distinguish homogenized from non-homogenized vectors.

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Homogeneous Others

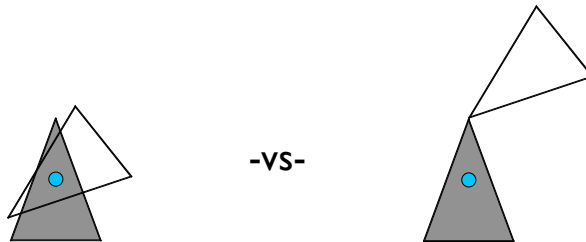
$$\tilde{\mathbf{A}} = \begin{bmatrix} & \mathbf{A} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now everything looks the same...
Hence the term “homogenized!”

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Compositing Matrices

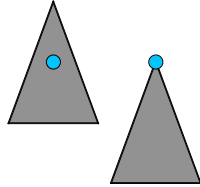
- Rotations and scales always about the origin
- How to rotate/scale about another point?



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Rotate About Arb. Point

- Step 1: Translate point to origin

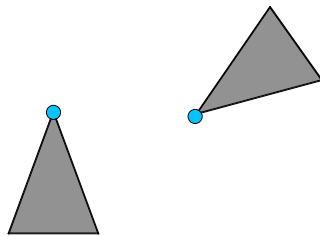


Translate $(-C)$

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Rotate About Arb. Point

- Step 1: Translate point to origin
- Step 2: Rotate as desired



Translate $(-C)$

Rotate (θ)

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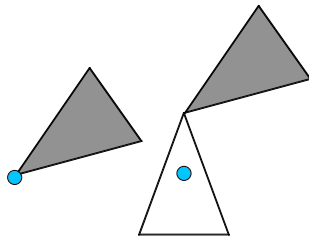
Rotate About Arb. Point

- Step 1: Translate point to origin
- Step 2: Rotate as desired
- Step 3: Put back where it was

Translate (-C)

Rotate (θ)

Translate (C)



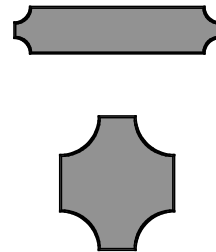
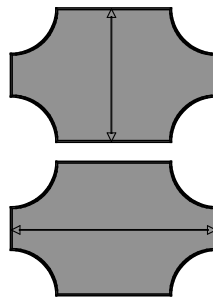
$$\tilde{\mathbf{p}}' = \underbrace{(-\mathbf{T})}_{\text{Don't negate the 1...}} \mathbf{R} \mathbf{T} \tilde{\mathbf{p}} = \mathbf{A} \tilde{\mathbf{p}}$$

Don't negate the 1...

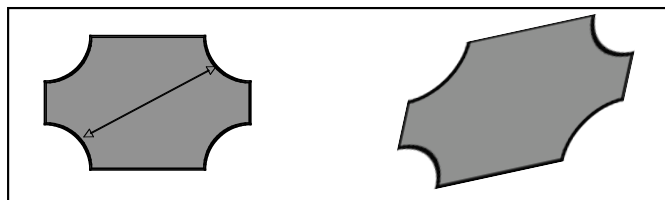
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Scale About Arb. Axis

- Diagonal matrices scale about coordinate axes only:



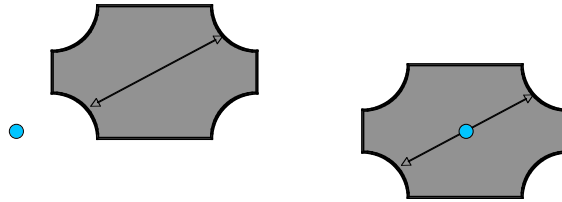
Not axis-aligned



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Scale About Arb.Axis

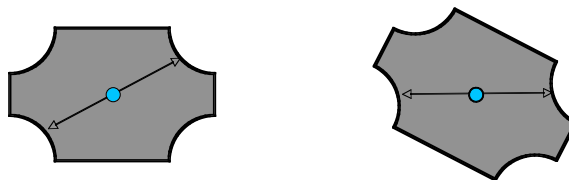
- Step 1: Translate axis to origin



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Scale About Arb.Axis

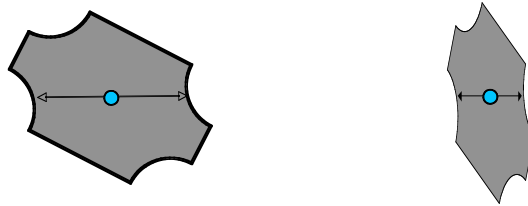
- Step 1: Translate axis to origin
- Step 2: Rotate axis to align with one of the coordinate axes



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Scale About Arb.Axis

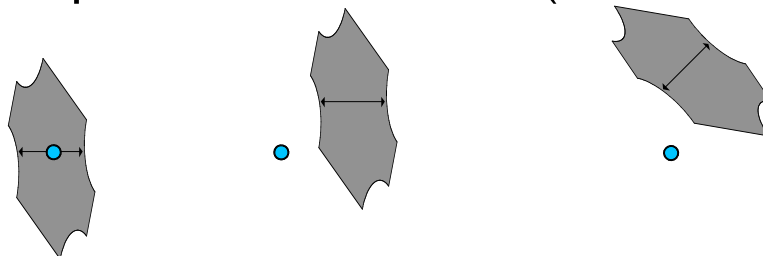
- Step 1: Translate axis to origin
- Step 2: Rotate axis to align with one of the coordinate axes
- Step 3: Scale as desired



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Scale About Arb.Axis

- Step 1: Translate axis to origin
- Step 2: Rotate axis to align with one of the coordinate axes
- Step 3: Scale as desired
- Steps 4&5: Undo 2 and 1 (reverse order)



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Order Matters!

- The order that matrices appear in matters

$$\mathbf{A} \cdot \mathbf{B} \neq \mathbf{BA}$$

- Some special cases work, but they are special
- But matrices are associative

$$(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$$

- Think about efficiency when you have many points to transform...

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Matrix Inverses

- In general: \mathbf{A}^{-1} undoes effect of \mathbf{A}

- Special cases:

- Translation: negate t_x and t_y
- Rotation: transpose
- Scale: invert diagonal (axis-aligned scales)

- Others:

- Invert matrix
- Invert SVD matrices

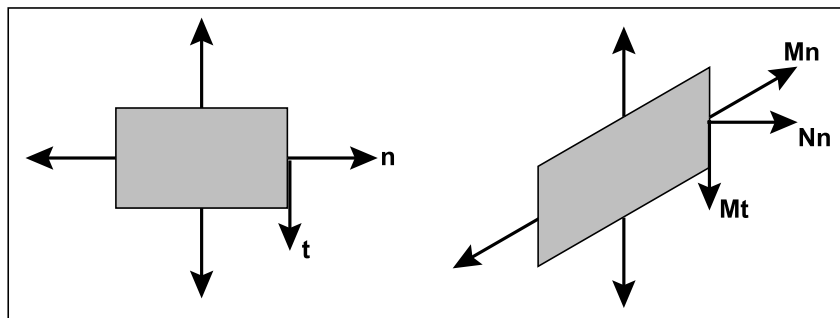
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Point Vectors / Direction Vectors

- Points in space have a 1 for the “ w ” coordinate
- What should we have for $\mathbf{a} - \mathbf{b}$?
 - $w = 0$
 - Directions not the same as positions
 - Difference of positions is a direction
 - Position + direction is a position
 - Direction + direction is a direction
 - Position + position is nonsense

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Somethings Require Care



For example normals do not transform normally

$$\mathbf{M}(\mathbf{a} \times \mathbf{b}) \neq (\mathbf{M}\mathbf{a}) \times (\mathbf{M}\mathbf{b})$$

$$\boxed{\mathbf{M}(\mathbf{R}\mathbf{e}) \neq \mathbf{R}(\mathbf{M}\mathbf{e})}$$

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