

CS-184: Computer Graphics

Lecture #14: Curves and Surfaces

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V2006S-14-1.0

Today

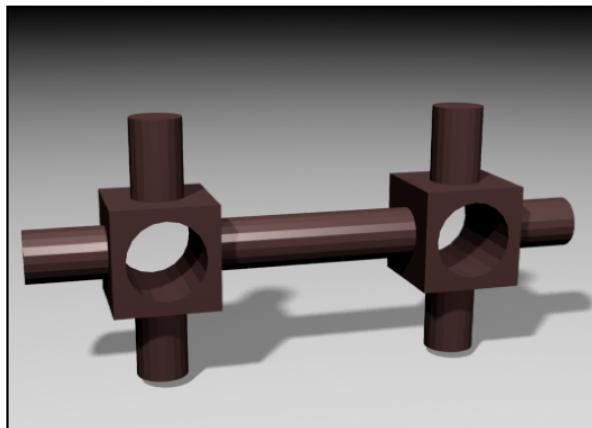
- General curve and surface representations
- Splines and other polynomial bases

Geometry Representations

- Constructive Solid Geometry (CSG)
- Parametric
 - Polygons
 - Subdivision surfaces
- Implicit Surfaces
- Point-based Surface
- Not always clear distinctions
 - *i.e.* CSG done with implicits

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Geometry Representations

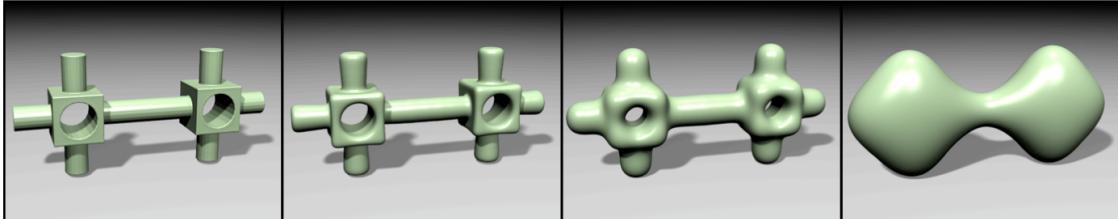


Object made by CSG
Converted to polygons

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Geometry Representations

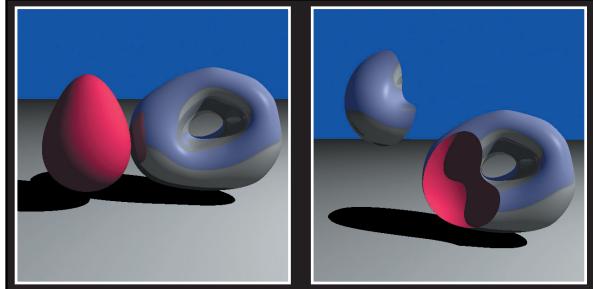
Object made by CSG
Converted to polygons
Converted to implicit surface



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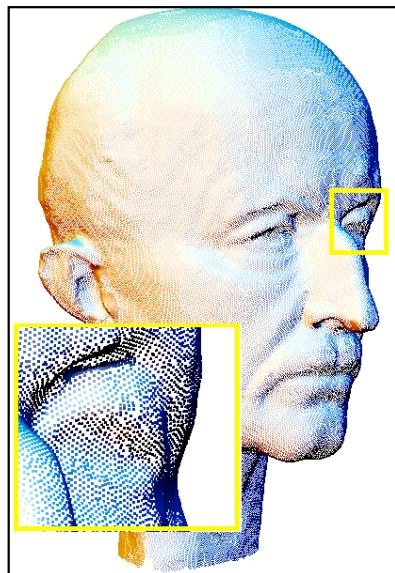
Geometry Representations

CSG on implicit surfaces

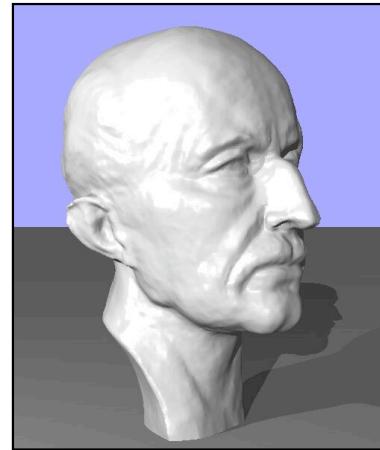


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Geometry Representations



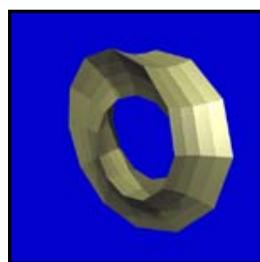
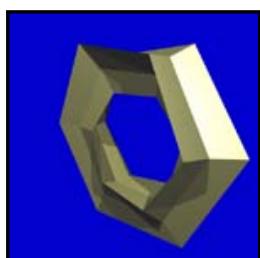
Point-based surface descriptions



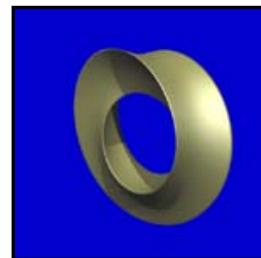
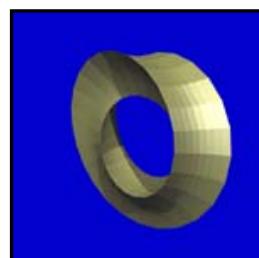
Ohtake, et al., SIGGRAPH 2003

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Geometry Representations



Subdivision surface
(different levels of refinement)



Images from Subdivision.org

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Geometry Representations

- Various strengths and weaknesses
 - Ease of use for design
 - Ease/speed for rendering
 - Simplicity
 - Smoothness
 - Collision detection
 - Flexibility (in more than one sense)
 - Suitability for simulation
 - *many others...*

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Parametric Representations

Curves: $\mathbf{x} = \mathbf{x}(u)$ $\mathbf{x} \in \mathbb{R}^n$ $u \in \mathbb{R}$

Surfaces: $\mathbf{x} = \mathbf{x}(u, v)$ $\mathbf{x} \in \mathbb{R}^n$ $u, v \in \mathbb{R}$
 $\mathbf{x} = \mathbf{x}(\mathbf{u})$ $\mathbf{u} \in \mathbb{R}^2$

Volumes: $\mathbf{x} = \mathbf{x}(u, v, w)$ $\mathbf{x} \in \mathbb{R}^n$ $u, v, w \in \mathbb{R}$
 $\mathbf{x} = \mathbf{x}(\mathbf{u})$ $\mathbf{u} \in \mathbb{R}^3$

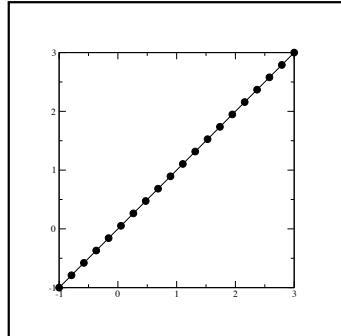
and so on...

Note: a vector function is really n scalar functions

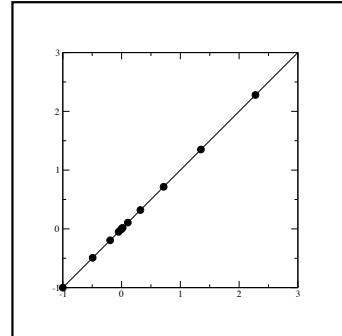
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Parametric Rep. Non-unique

- Same curve/surface may have multiple formulae



$$\mathbf{x}(u) = [u, u]$$



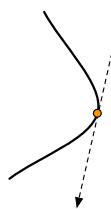
$$\mathbf{x}(u) = [u^3, u^3]$$

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Simple Differential Geometry

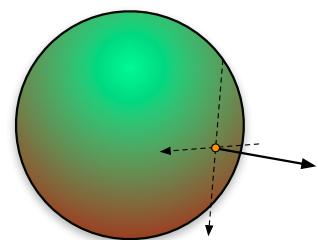
- Tangent to curve

$$\mathbf{t}(u) = \left. \frac{\partial \mathbf{x}}{\partial u} \right|_u$$



- Tangents to surface

$$\mathbf{t}_u(u, v) = \left. \frac{\partial \mathbf{x}}{\partial u} \right|_{u,v} \quad \mathbf{t}_v(u, v) = \left. \frac{\partial \mathbf{x}}{\partial v} \right|_{u,v}$$



- Normal of surface

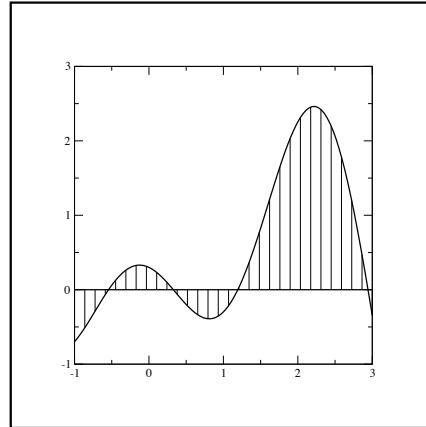
$$\hat{\mathbf{n}} = \frac{\mathbf{t}_u \times \mathbf{t}_v}{\|\mathbf{t}_u \times \mathbf{t}_v\|}$$

- Also: curvature, curve normals, curve bi-normal, others...
- Degeneracies: $\partial \mathbf{x} / \partial u = 0$ or $\mathbf{t}_u \times \mathbf{t}_v = 0$

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Discretization

- Arbitrary curves have an uncountable number of parameters



i.e. specify function value at all points
on real number line

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Discretization

- Arbitrary curves have an uncountable number of parameters

- Pick *complete* set of basis functions

- Polynomials, Fourier series, etc.

$$x(u) = \sum_{i=0}^{\infty} c_i \phi_i(u)$$

- Truncate set at some reasonable point

$$x(u) = \sum_{i=0}^3 c_i \phi_i(u) = \sum_{i=0}^3 c_i u^i$$

- Function represented by the vector (list) of c_i

- The c_i may themselves be vectors

$$\mathbf{x}(u) = \sum_{i=0}^3 \mathbf{c}_i \phi_i(u)$$

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Polynomial Basis

◦ Power Basis

$$x(u) = \sum_{i=0}^d c_i u^i$$

$$x(u) = \mathbf{C} \cdot \mathcal{P}^d$$

$$\mathbf{C} = [c_0, c_1, c_2, \dots, c_d]$$

$$\mathcal{P}^d = [1, u, u^2, \dots, u^d]$$

The elements of \mathcal{P}^d are *linearly independant*
i.e. no good approximation

$$u^k \not\approx \sum_{i \neq k} c_i u^i$$

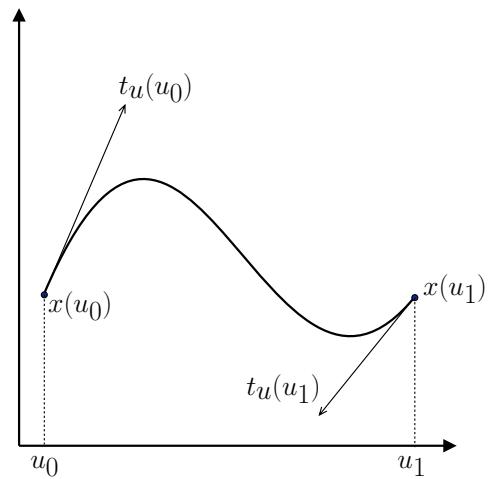
Skipping something would lead to bad results... odd stiffness

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Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?

For now, assume
 $u_0 = 0$ $u_1 = 1$



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Specifying a Curve

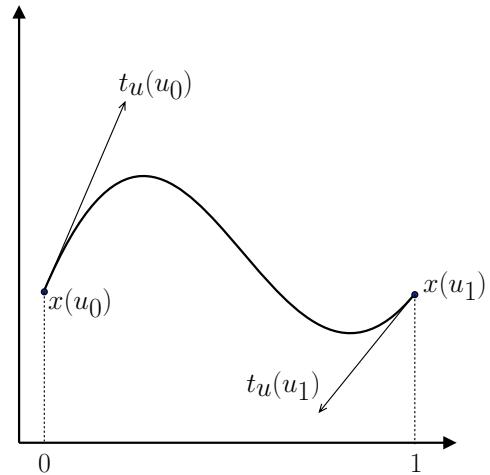
Given desired values (constraints) how do we determine the coefficients for cubic power basis?

$$x(0) = c_0 = x_0$$

$$x(1) = \Sigma c_i = x_1$$

$$x'(0) = c_1 = x'_0$$

$$x'(1) = \Sigma i c_i = x'_1$$



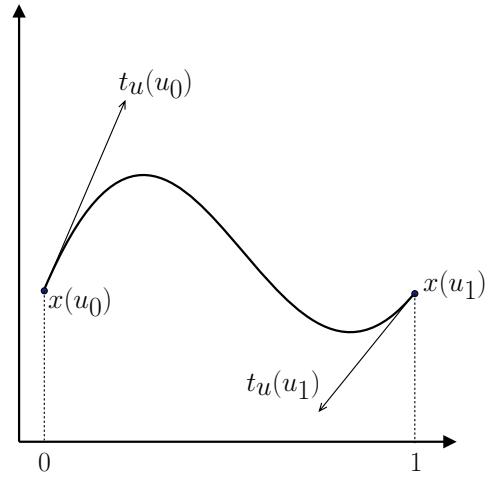
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Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?

$$\begin{bmatrix} x_0 \\ x_1 \\ x'_0 \\ x'_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\mathbf{p} = \mathbf{B} \cdot \mathbf{c}$$



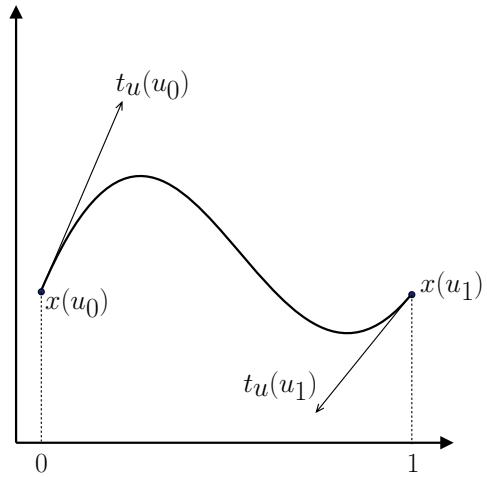
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Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?

$$\mathbf{c} = \beta_{\text{H}} \cdot \mathbf{p}$$

$$\beta_{\text{H}} = \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & 1 \\ 2 & -2 & 1 & 1 \end{bmatrix}$$



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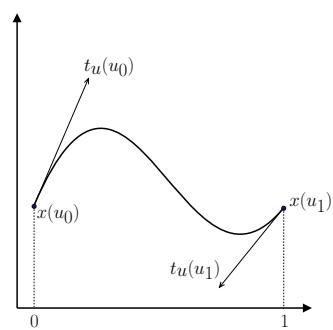
Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?

$$\mathbf{c} = \beta_{\text{H}} \cdot \mathbf{p}$$

$$x(u) = \mathcal{P}^3 \cdot \mathbf{c} = \boxed{\mathcal{P}^3 \beta_{\text{H}}} \mathbf{p}$$

$$= \begin{bmatrix} 1 + 0u - 3u^2 + 2u^3 \\ 0 + 0u + 3u^2 - 2u^3 \\ 0 + 1u - 2u^2 + 1u^3 \\ 0 + 0u - 1u^2 + 1u^3 \end{bmatrix} \mathbf{p}$$



$$\beta_{\text{H}} = \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & 1 \\ 2 & -2 & 1 & 1 \end{bmatrix}$$

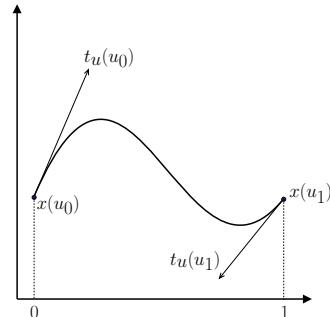
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Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?

$$\mathbf{c} = \beta_H \cdot \mathbf{p}$$

$$x(u) = \begin{bmatrix} 1 + 0u - 3u^2 + 2u^3 \\ 0 + 0u + 3u^2 - 2u^3 \\ 0 + 1u - 2u^2 + 1u^3 \\ 0 + 0u - 1u^2 + 1u^3 \end{bmatrix} \mathbf{p}$$



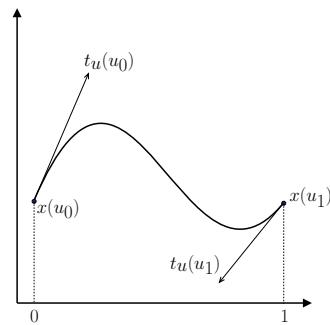
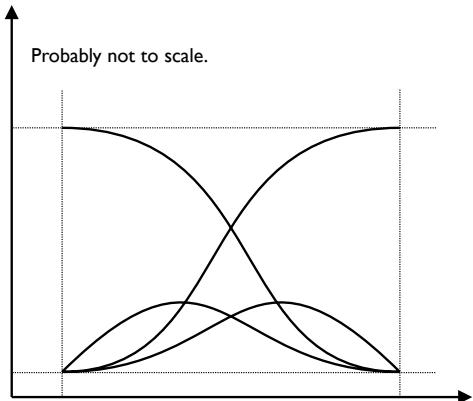
$$x(u) = \sum_{i=0}^3 p_i b_i(u)$$

Hermite basis functions

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Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?



$$x(u) = \sum_{i=0}^3 p_i b_i(u)$$

Hermite basis functions

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Hermite Basis

- Specify curve by
 - Endpoint values
 - Endpoint tangents (derivatives)
- Parameter interval is arbitrary (most times)
 - Don't need to recompute basis functions
- These are *cubic* Hermite
 - Could do construction for any odd degree
 - $(d - 1)/2$ derivatives at end points

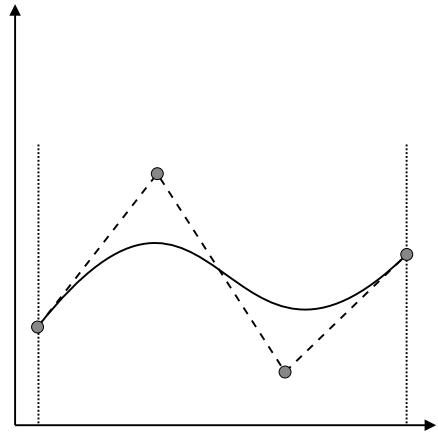
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Cubic Bézier

- Similar to Hermite, but specify tangents indirectly

$$\begin{aligned}x_0 &= p_0 \\x_1 &= p_3 \\x'_0 &= 3(p_1 - p_0) \\x'_1 &= 3(p_3 - p_2)\end{aligned}$$

Note: all the control points are points in space, no tangents.



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Cubic Bézier

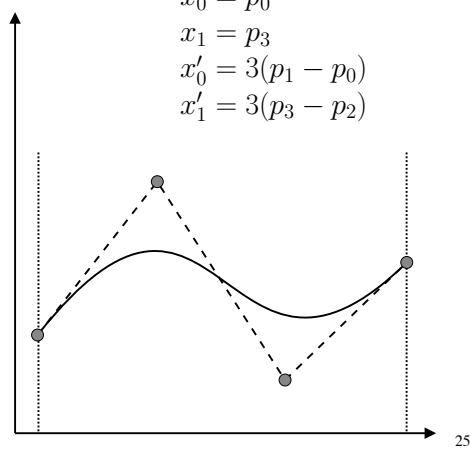
- Similar to Hermite, but specify tangents indirectly

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \mathbf{p}$$

$$\mathbf{c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \mathbf{p}$$

$$\mathbf{c} = \beta_z \mathbf{p}$$

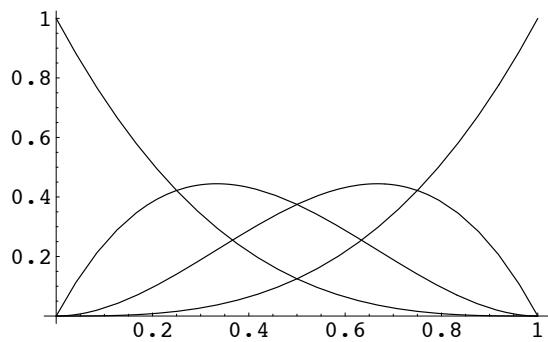
$$\begin{aligned} x_0 &= p_0 \\ x_1 &= p_3 \\ x'_0 &= 3(p_1 - p_0) \\ x'_1 &= 3(p_3 - p_2) \end{aligned}$$



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Cubic Bézier

- Plot of Bézier basis functions



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Changing Bases

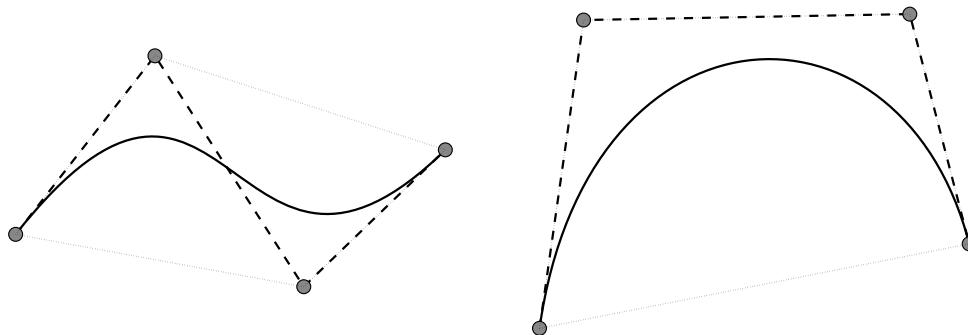
- Power basis, Hermite, and Bézier all are still just cubic polynomials
 - The three basis sets all span the same space
 - Like different axes in \mathbb{R}^3
- Changing basis

$$\begin{aligned}\mathbf{c} &= \beta_z \mathbf{p}_z & \mathbf{p}_z &= \beta_z^{-1} \beta_h \mathbf{p}_h \\ \mathbf{c} &= \beta_h \mathbf{p}_h\end{aligned}$$

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Useful Properties of a Basis

- Convex Hull
 - All points on curve inside convex hull of control points
 - $\sum_i b_i(u) = 1$ $b_i(u) \geq 0$ $\forall u \in \Omega$
 - Bézier basis has convex hull property



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Useful Properties of a Basis

- Invariance under class of transforms
 - Transforming curve is same as transforming control points
 - $\mathbf{x}(u) = \sum_i \mathbf{p}_i b_i(u) \Leftrightarrow T\mathbf{x}(u) = \sum_i (T\mathbf{p}_i) b_i(u)$
 - Bézier basis invariant for affine transforms
 - Bézier basis NOT invariant for perspective transforms
 - NURBS are though...

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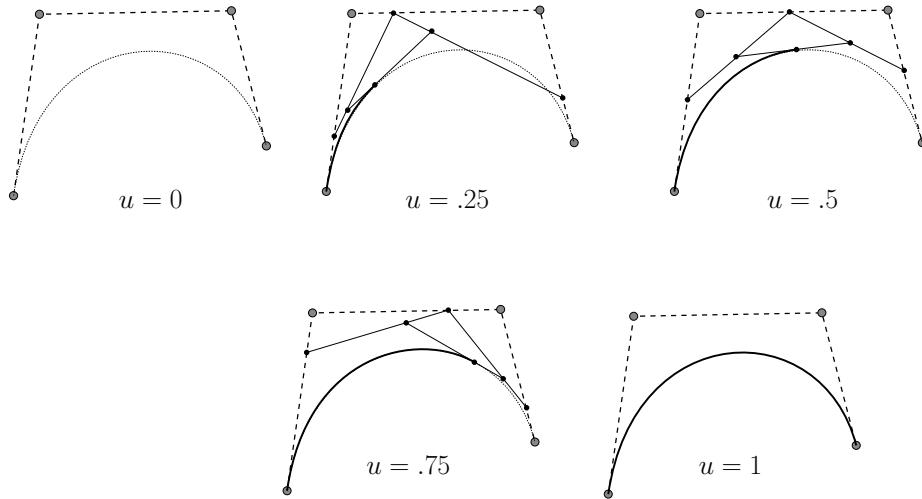
Useful Properties of a Basis

- Local support
 - Changing one control point has limited impact on entire curve
 - Nice subdivision rules
 - Orthogonality ($\int_{\Omega} b_i(u)b_j(u)du = \delta_{ij}$)
 - Fast evaluation scheme
 - Interpolation -vs- approximation

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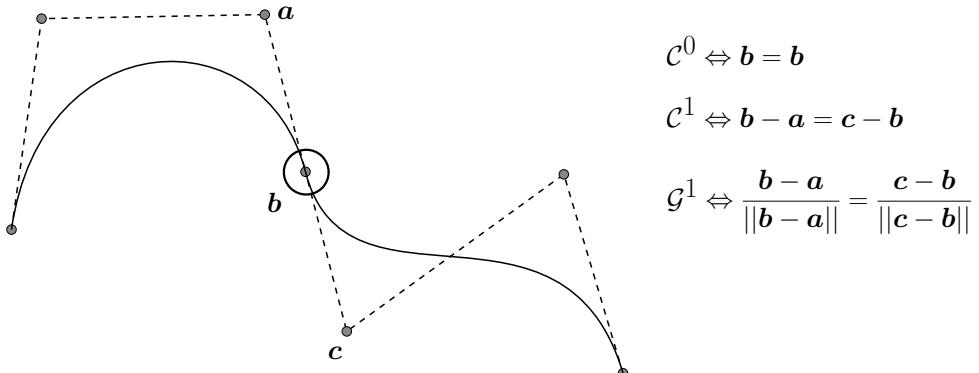
DeCasteljau Evaluation

- A geometric evaluation scheme for Bézier



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Joining



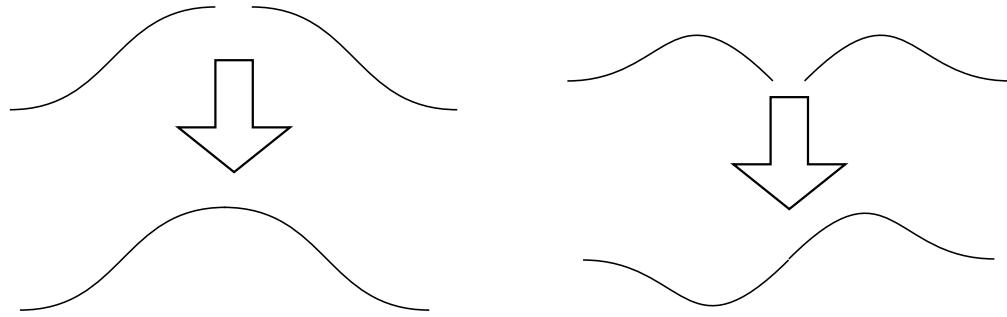
If you change a , b , or c you must change the others

But if you change a , b , or c you do not have to change beyond those three. *LOCAL SUPPORT*

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“Hump” Functions

- Constraints at joining can be built in to make new basis



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Tensor-Product Surfaces

- Surface is a curve swept through space
- Replace control points of curve with other curves

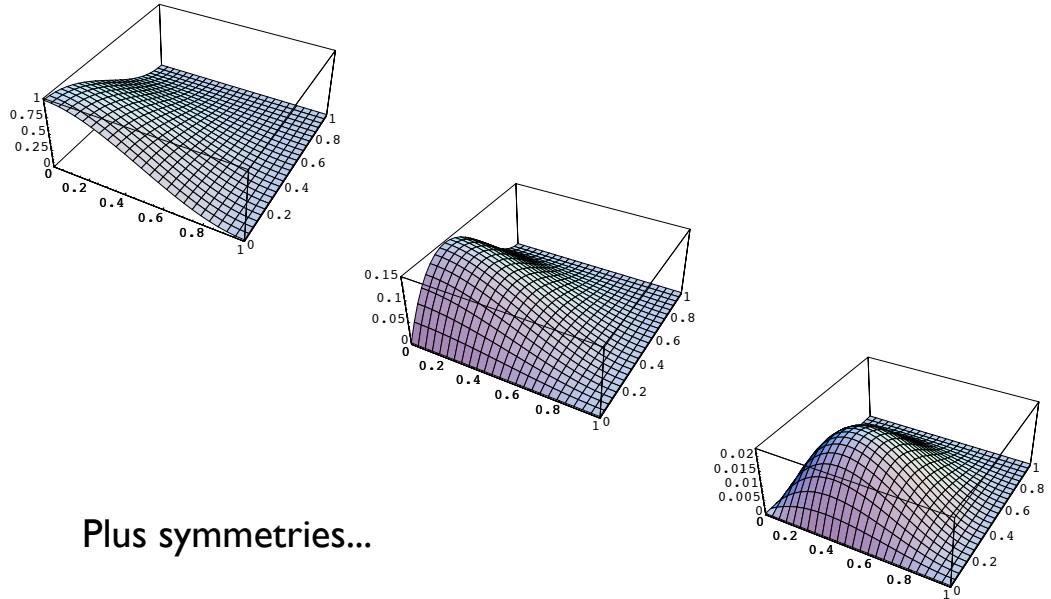
$$x(u, v) = \sum_i p_i b_i(u) \\ \sum_i q_i(v) b_i(u) \qquad \qquad q_i(v) = \sum_j p_{ji} b_j(v)$$

$$x(u, v) = \sum_{ij} p_{ij} b_i(u) b_j(v) \qquad b_{ij}(u, v) = b_i(u) b_j(v)$$

$$x(u, v) = \sum_{ij} p_{ij} b_{ij}(u, v)$$

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Hermite Surface Bases



Hermite Surface Hump Functions

