CS-184: Computer Graphics
Lecture #21: Fluid Simulation II

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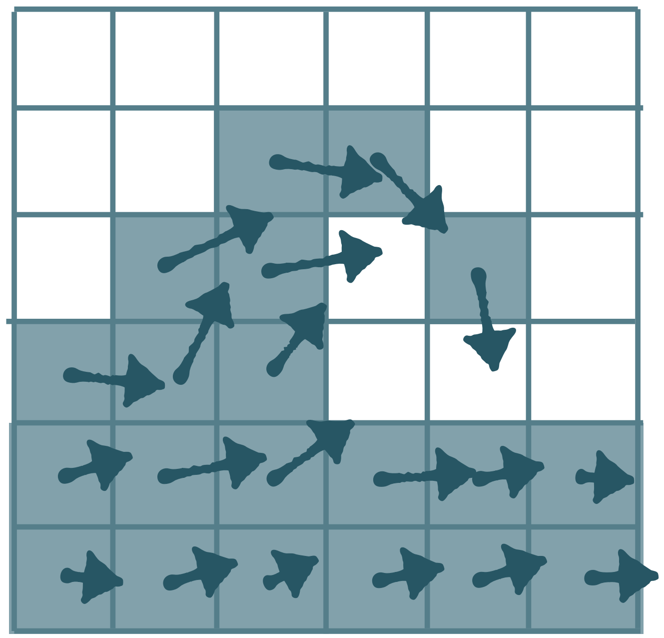
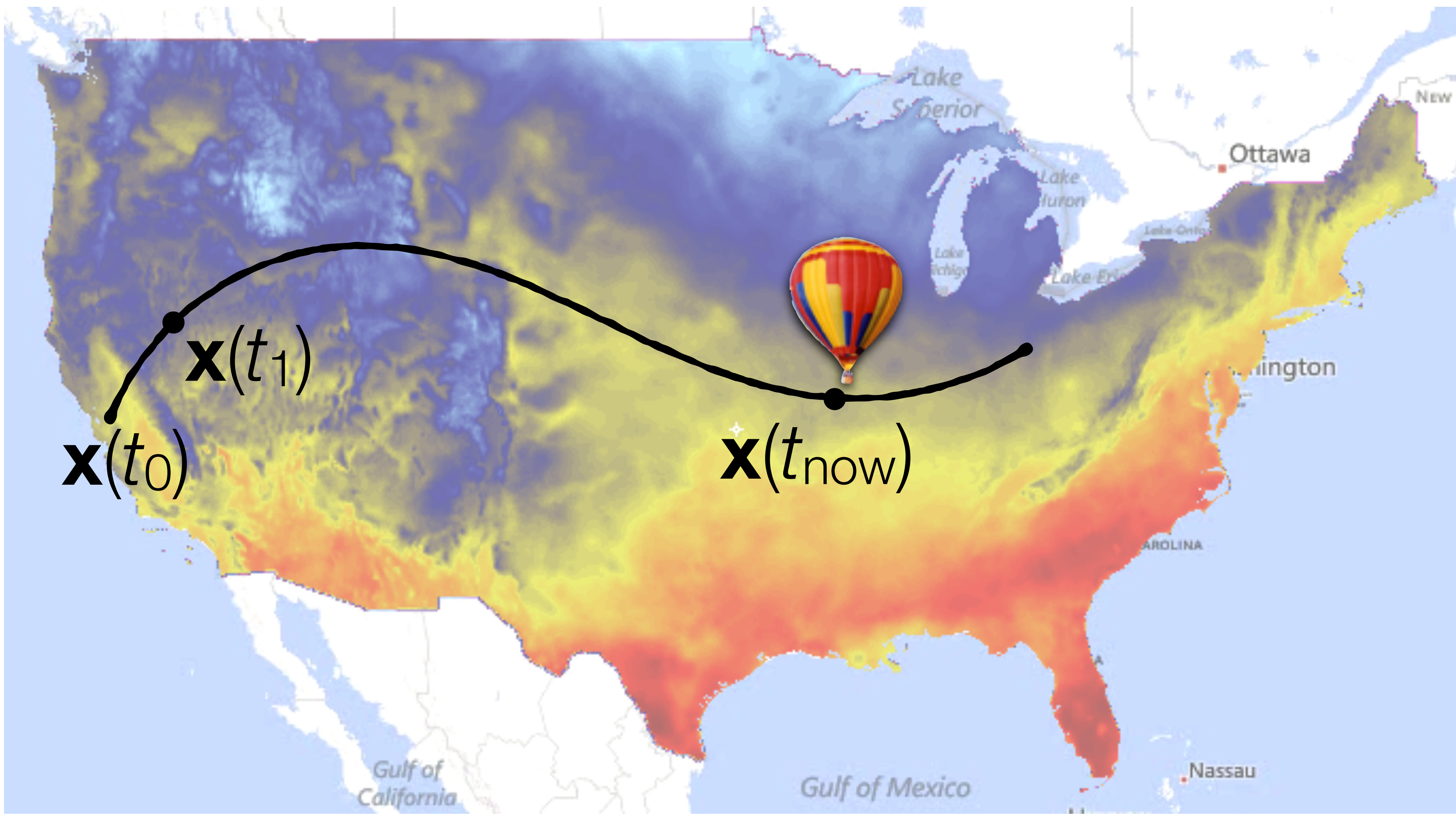
Grid-based fluid simulation

- Recap: **Eulerian** viewpoint
  - Grid is fixed, fluid moves through it
- How does the velocity at a grid cell change over time?



Eulerian and Lagrangian time derivatives

- Consider a weather balloon moving with the wind, measuring air temperature $T(\mathbf{x}, t)$



Eulerian and Lagrangian time derivatives

- Consider a weather balloon moving with the wind. It measures air temperature as $T(\mathbf{x}(t), t)$

$$\frac{\mathrm{d}}{\mathrm{d}t} T\big(\mathbf{x}(t), t\big) = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial \mathbf{x}} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$$

[Chain rule]

$$= \frac{\partial T}{\partial t} + \nabla T \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$$

$$= \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T$$

Temperature as seen by balloon

Velocity of balloon = velocity of air

Eulerian and Lagrangian time derivatives

$$\frac{\mathrm{D}T}{\mathrm{D}t} = \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T$$

Lagrangian derivative:
change seen by a point moving with the fluid

Change due to movement of fluid ("**advection**")

Eulerian derivative:
change at a fixed point

The fluid equations

- Newton's second law

$$\mathbf{a} = \mathbf{f}/\rho$$

- Forces:

$$\mathbf{f} = \mathbf{f}^{\text{ext}} - \nabla p + \mu \nabla^2 \mathbf{u}$$

Reminder:
Velocity is **u** now

- Acceleration is **Lagrangian**:

$$\mathbf{a} = \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}$$

Acceleration of fluid "molecule"

Change at a grid cell

Flow advects itself!

The fluid equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{\rho}\left(\mathbf{f}^{\text{ext}} - \nabla p + \mu \nabla^2 \mathbf{u}\right)$$

$$p = ?$$

- The **Navier-Stokes equations**

C.-L. Navier

G.G. Stokes

Millenium prize:
$1,000,000 to prove
(or disprove)
existence & smoothness
of solutions

Operator splitting

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{\rho}\left(\mathbf{f}^{\text{ext}} - \nabla p + \mu \nabla^2 \mathbf{u}\right)$$

- Lots of different terms; hard to integrate in one go
- Deal with one term at a time
  - (ignoring all the others)

Operator splitting

$\mathbf{u}^n \rightarrow$

Integrate $\dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = 0$ for time Δt — Advection

$\downarrow$

Integrate $\dfrac{\partial \mathbf{u}}{\partial t} = \dfrac{1}{\rho} \mathbf{f}^{\mathrm{ext}}$ for time Δt — External forces

$\downarrow$

Viscosity — Integrate $\dfrac{\partial \mathbf{u}}{\partial t} = \dfrac{\mu}{\rho} \nabla^2 \mathbf{u}$ for time Δt

$\downarrow$

Pressure — Integrate $\dfrac{\partial \mathbf{u}}{\partial t} = -\dfrac{1}{\rho} \nabla p$ for time Δt $\rightarrow \mathbf{u}^{n+1}$

Advection

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = 0$$

- Transport a quantity (in this case, **u**) via the velocity field **u**

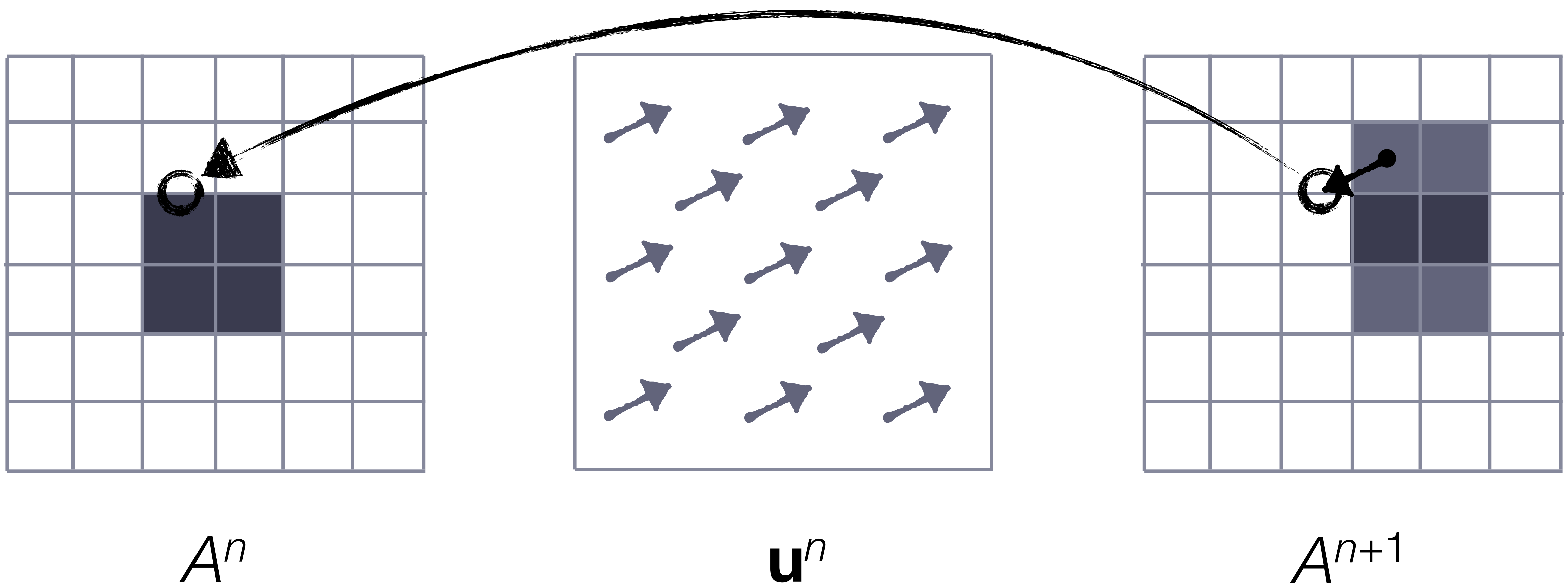
- Confusing! Let's transport something else first

$$\frac{\partial A}{\partial t} + \mathbf{u} \cdot \nabla A = 0$$

- *e.g.* colour, temperature, concentration of ink in water / smoke in air / *etc.*

Advection

$$\frac{\partial A}{\partial t} + \mathbf{u} \cdot \nabla A = 0$$



Transport A by tracing **backwards** and looking up its value

Advection

- Input: initial grid A^n, velocity field $\mathbf{u}^n$
- Output: final grid A^{n+1}
- For each grid cell $\mathbf{x}_i$
  - Backtrace position, e.g. $\mathbf{x}^{\text{back}} = \mathbf{x}_i - \mathbf{u}_i \Delta t$
  - Set output $A_i^{n+1} = \text{interpolate } A^n \text{ at } \mathbf{x}^{\text{back}}$

To advect velocities, just feed in $\mathbf{u}^n$ as the initial grid too.

External forces

- Gravity
- Buoyancy

Buoyancy strength

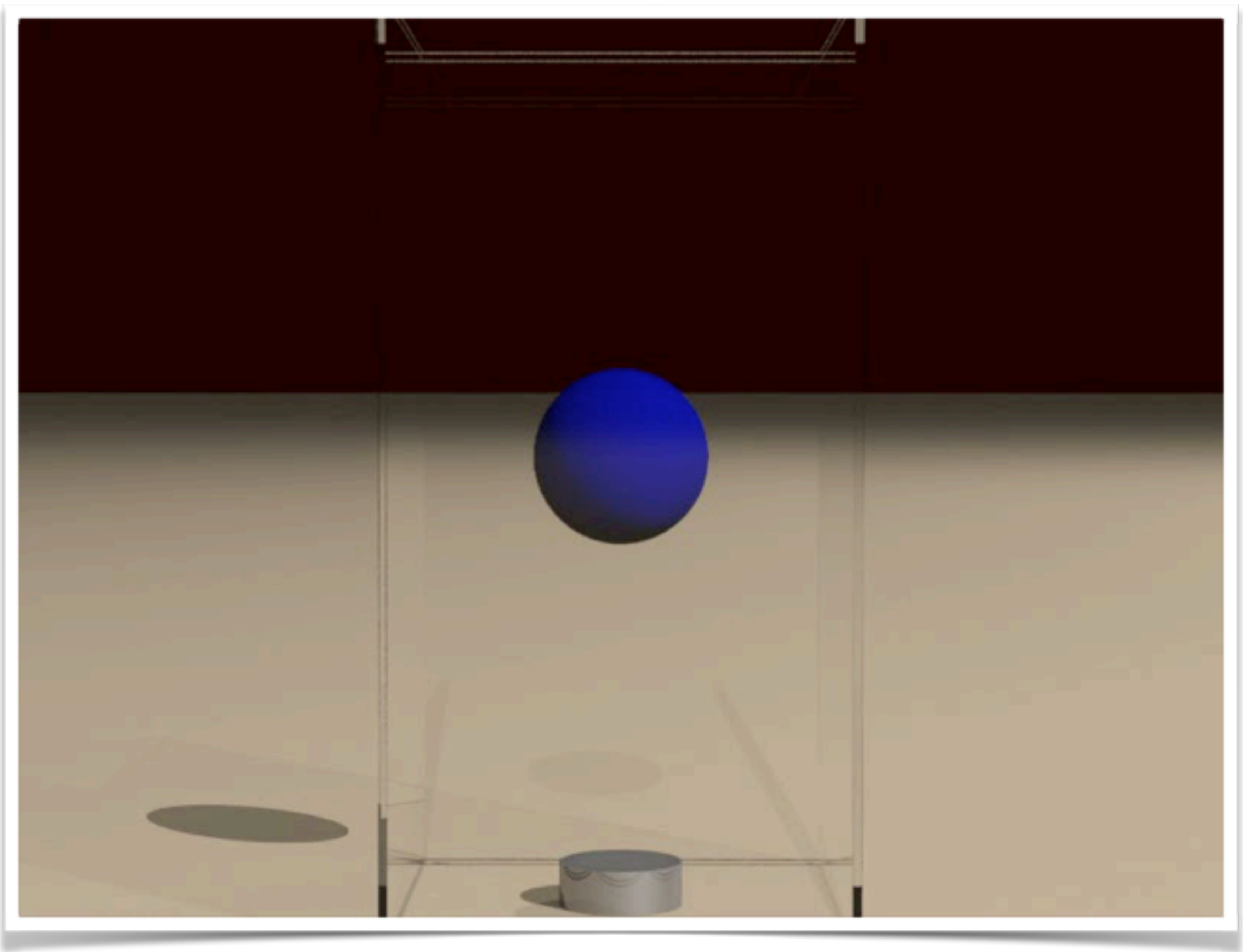
$$\mathbf{f}_{\text{ext}} = -\beta(T - T_{\text{amb}})\mathbf{g}$$

Temperature of fluid

"Ambient" temperature

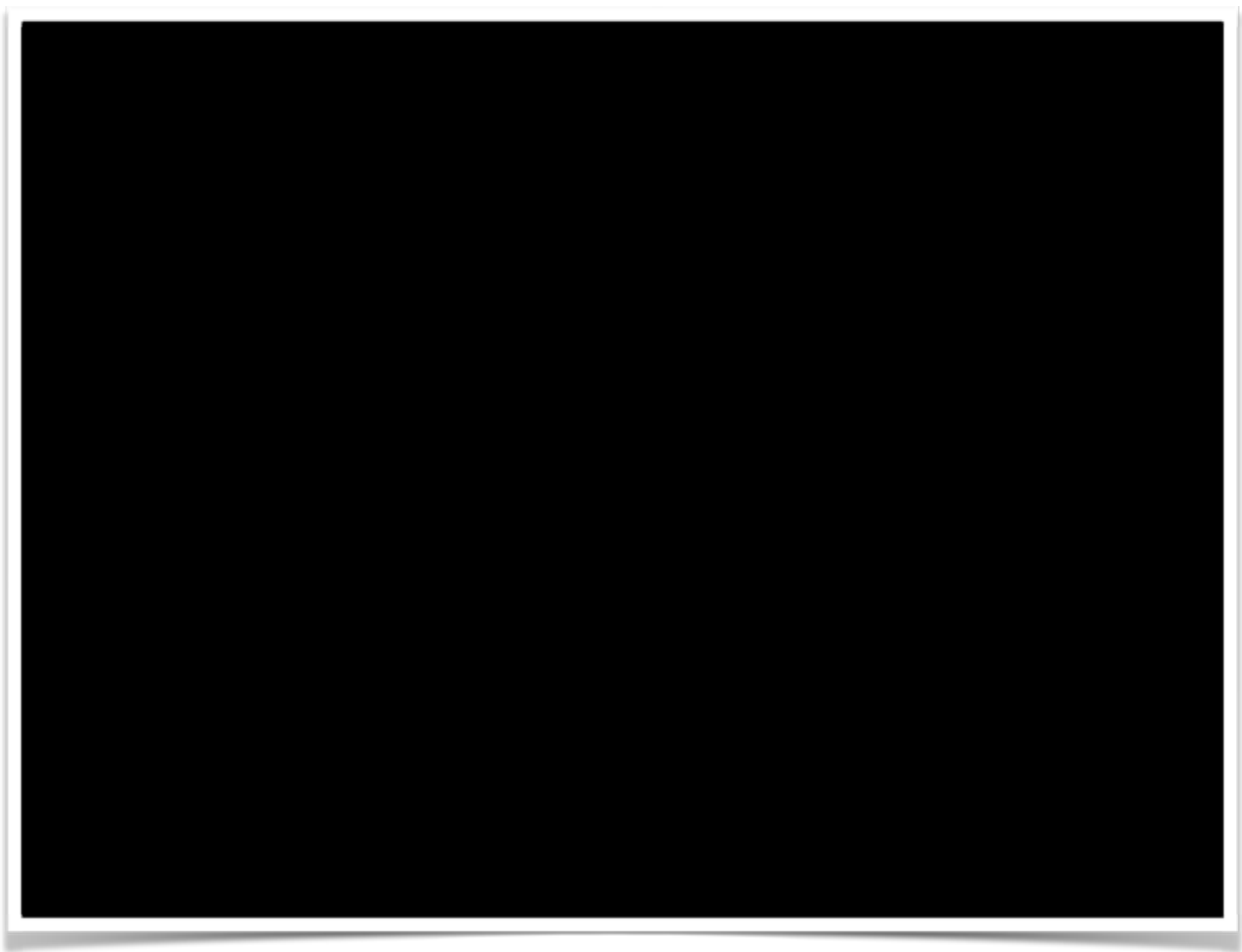
- User interaction

External forces



Lentine, Zheng, and Fedkiw, 2010

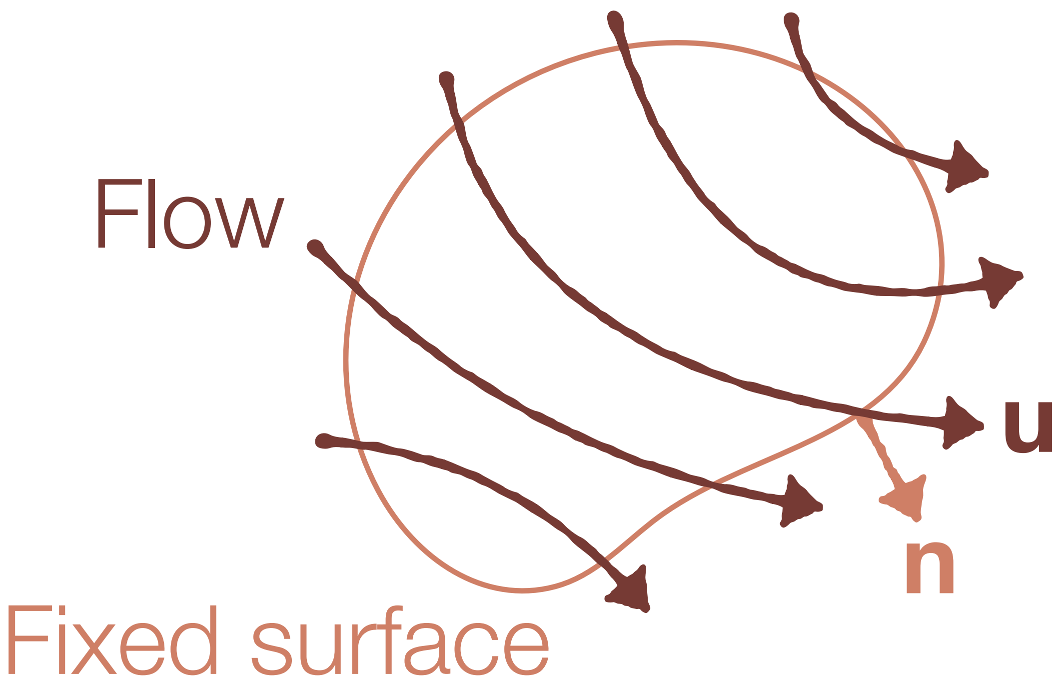
Pressure



Becker and Teschner, 2007

Incompressibility

- We will **prohibit** compressibility from our simulation.



Net flow into/out of region

$$= \iint \mathbf{u} \cdot \hat{\mathbf{n}} \, \mathrm{d}A$$

$$= \iiint \nabla \cdot \mathbf{u} \, \mathrm{d}V$$

[Divergence theorem]

We want net flow to be 0 for all possible regions, so…

$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ everywhere

Pressure

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho}\nabla p$$

$$p = ?$$

- Let's just do forward Euler…

Pressure

$$\mathbf{u}^{\text{new}} = \mathbf{u} - \frac{\Delta t}{\rho} \nabla p$$

$$\nabla \cdot \mathbf{u}^{\text{new}} = 0$$

- Let’s just do forward Euler…
- …and **choose** the *p* which makes $\mathbf{u}^{\text{new}}$ divergence-free

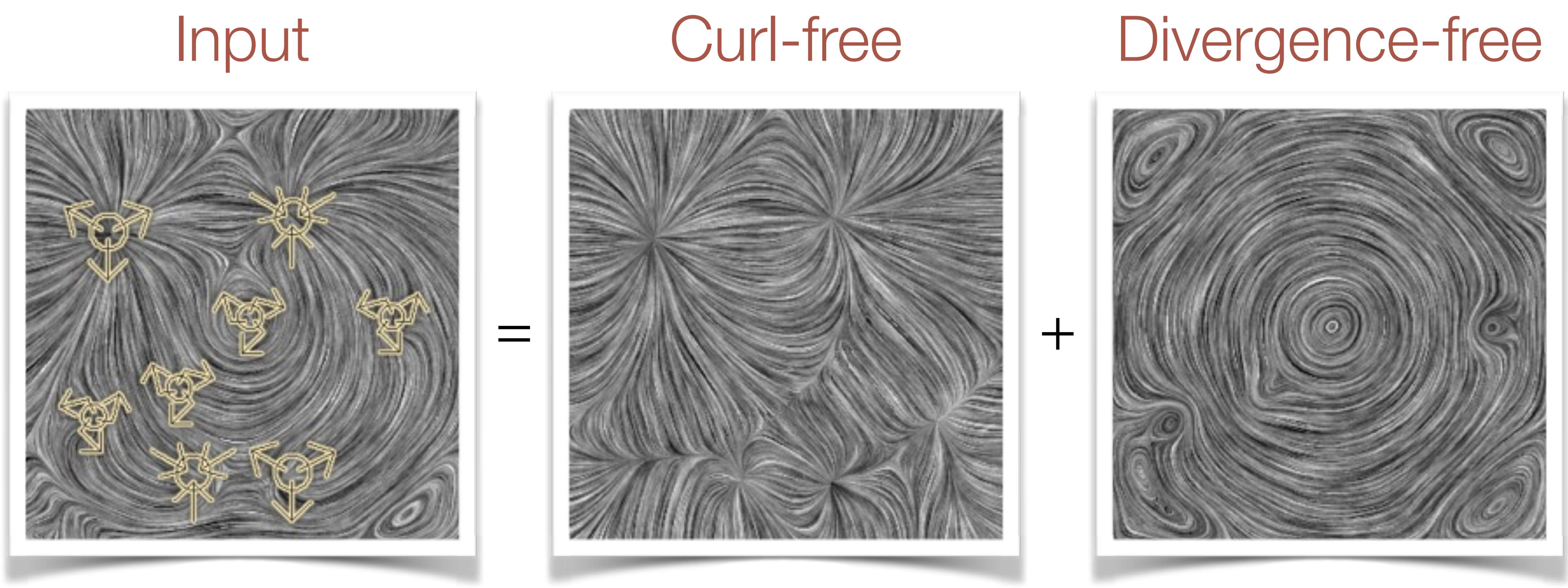
$$\nabla \cdot \mathbf{u} - \frac{\Delta t}{\rho} \nabla^2 p = 0$$

$$\nabla^2 p = \cancel{\frac{\rho}{\Delta t}} \nabla \cdot \mathbf{u}$$

Ignore the $\rho/\Delta t$
(just a rescaling)

The Helmholtz-Hodge decomposition

- Equivalently: separate **u** into **curl-free** and **divergence-free** components
  …then throw away the curl-free part



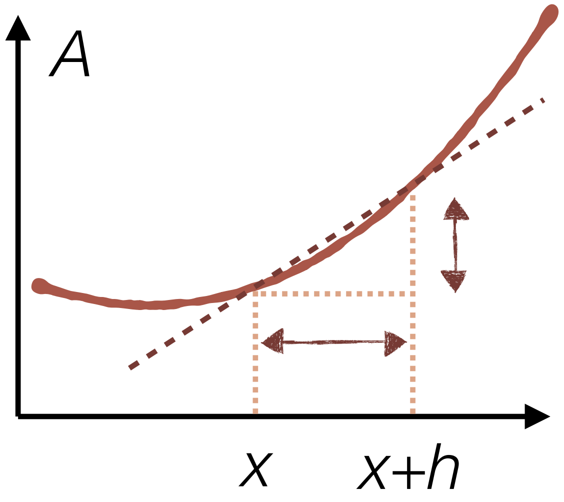
$$\mathbf{u} \quad - \quad \nabla p \quad = \quad \mathbf{u}^{\text{new}}$$

Pressure

- We just have to solve $\nabla^2 p = \nabla \cdot \mathbf{u}$
  - How?
- **Q1:** How to evaluate $\boldsymbol{\nabla}\cdot$ and $\boldsymbol{\nabla}^2$ on a grid
- **Q2:** How to store p and $\mathbf{u}$ on the grid in the first place

Finite differences in 1D

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$



- On a grid, we only have samples at grid spacing Δx

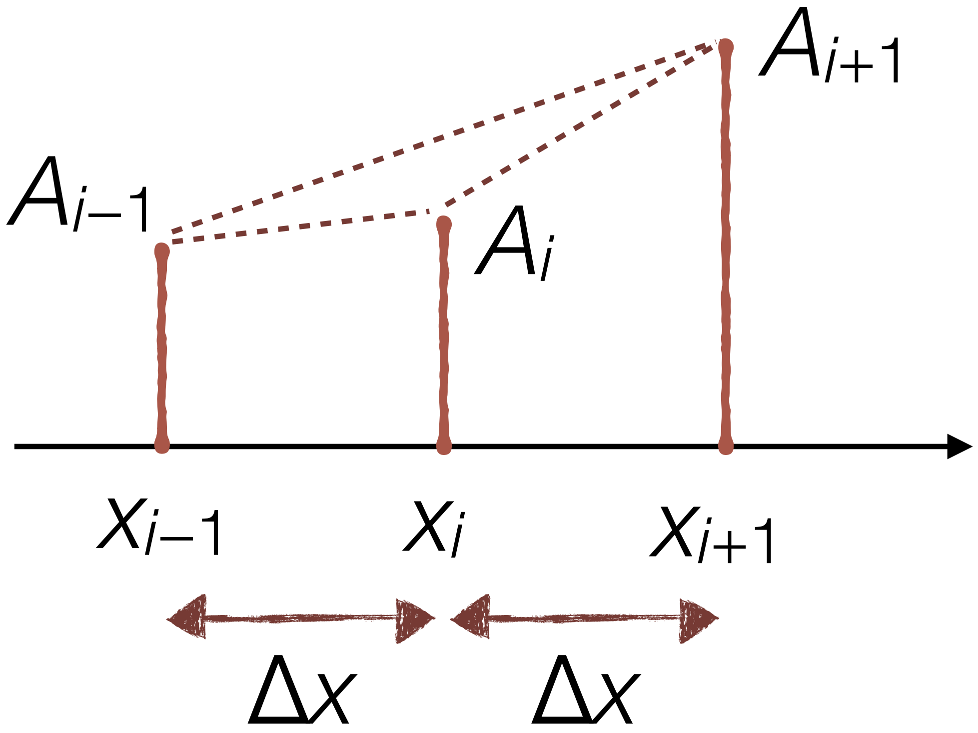
$$A'_i \approx \frac{A_{i+1} - A_i}{\Delta x}$$

or

$$A'_i \approx \frac{A_i - A_{i-1}}{\Delta x}$$

or

$$A'_i \approx \frac{A_{i+1} - A_{i-1}}{2\Delta x}$$



Finite differences in 2D

$$\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}$$

	+1	
+1	-4	+1
	+1	

- Apply forward and backward differences

$$\left(\frac{\partial^2 A}{\partial x^2}\right)_{i,j} \approx \frac{A_{i-1,j} - 2A_{i,j} + A_{i+1,j}}{\Delta x^2}$$

+1	-2	+1

$$\left(\frac{\partial^2 A}{\partial y^2}\right)_{i,j} \approx \frac{A_{i,j-1} - 2A_{i,j} + A_{i,j+1}}{\Delta x^2}$$

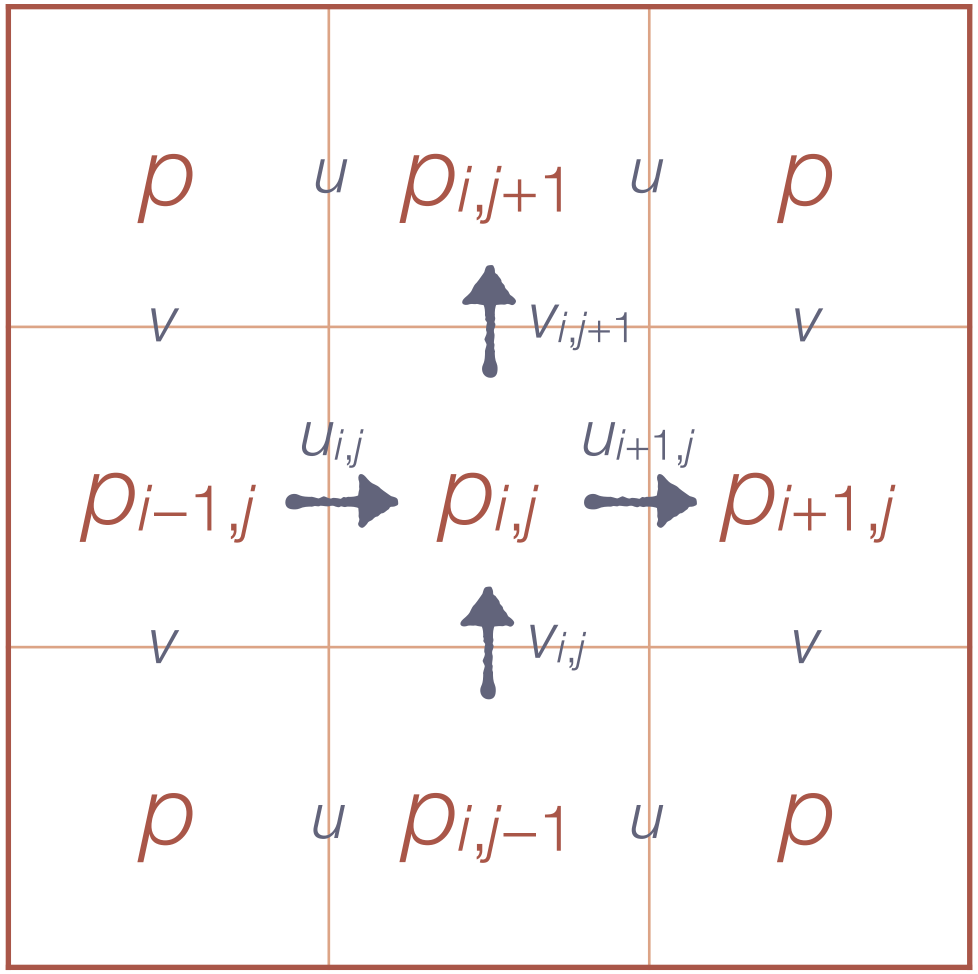
+1
-2
+1

Five point **stencil** for the Laplacian

Staggered (marker-and-cell) grids

- Store pressure at cell centers, but velocity at cell **faces**

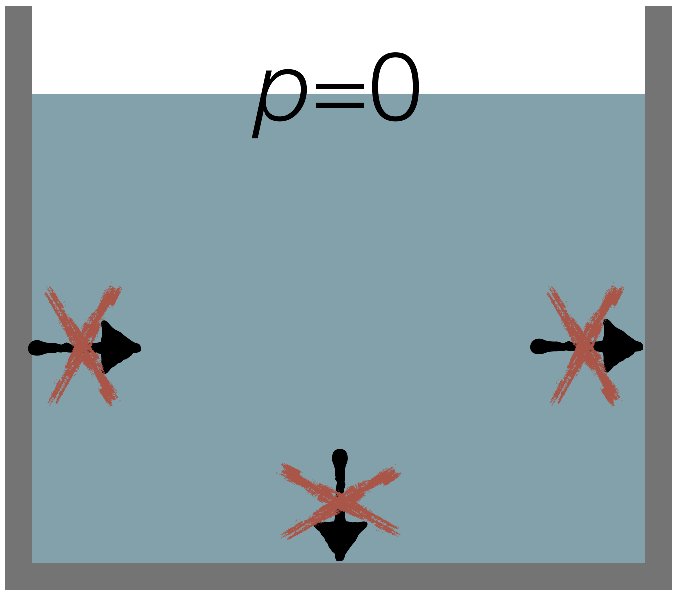
$$(\nabla \cdot \mathbf{u})_{i,j} \approx \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + \frac{v_{i,j+1} - v_{i,j}}{\Delta x}$$



- Finite differences line up
  - $\nabla \cdot \mathbf{u}$ and p at cell centers
  - Components of ∇p and $\mathbf{u}$ at cell faces

Boundary conditions

- At solid obstacles, $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$
    - Fluid cannot flow into or out of obstacles
- At free surface, $p = 0$
    - Air applies negligible force on water



Pressure solve

- Build a linear system with one equation per cell

$$(\nabla^2 p)_{i,j} = (\nabla \cdot \mathbf{u})_{i,j}$$

- Whole system: $\mathbf{Ax} = \mathbf{b}$, where

  - $\mathbf{x}$ is a vector containing all the $p_{i,j}$
    $\mathbf{b}$ is a vector containing $(\nabla \cdot \mathbf{u})_{i,j}$
    Rows of $\mathbf{A}$ contain stencil for ∇^2

- Be careful about boundaries!

Pressure solve

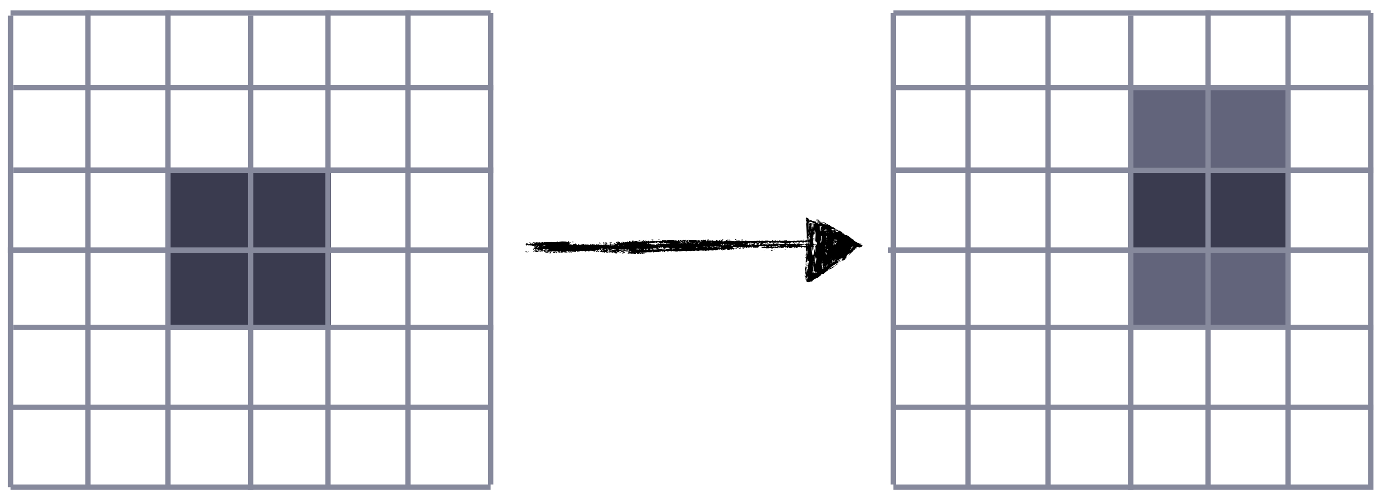
- Solve $\mathbf{Ax} = \mathbf{b}$ to get pressure values p
    - $\mathbf{A}$ is large, sparse, symmetric, positive (semi)definite
    - Use *e.g.* preconditioned conjugate gradient method
- Update velocities: $\mathbf{u}^{\text{new}} = \mathbf{u} - \nabla p$

Refer to Bridson & Müller-Fischer 2007 for full details

Viscosity

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\mu}{\rho} \nabla^2 \mathbf{u}$$

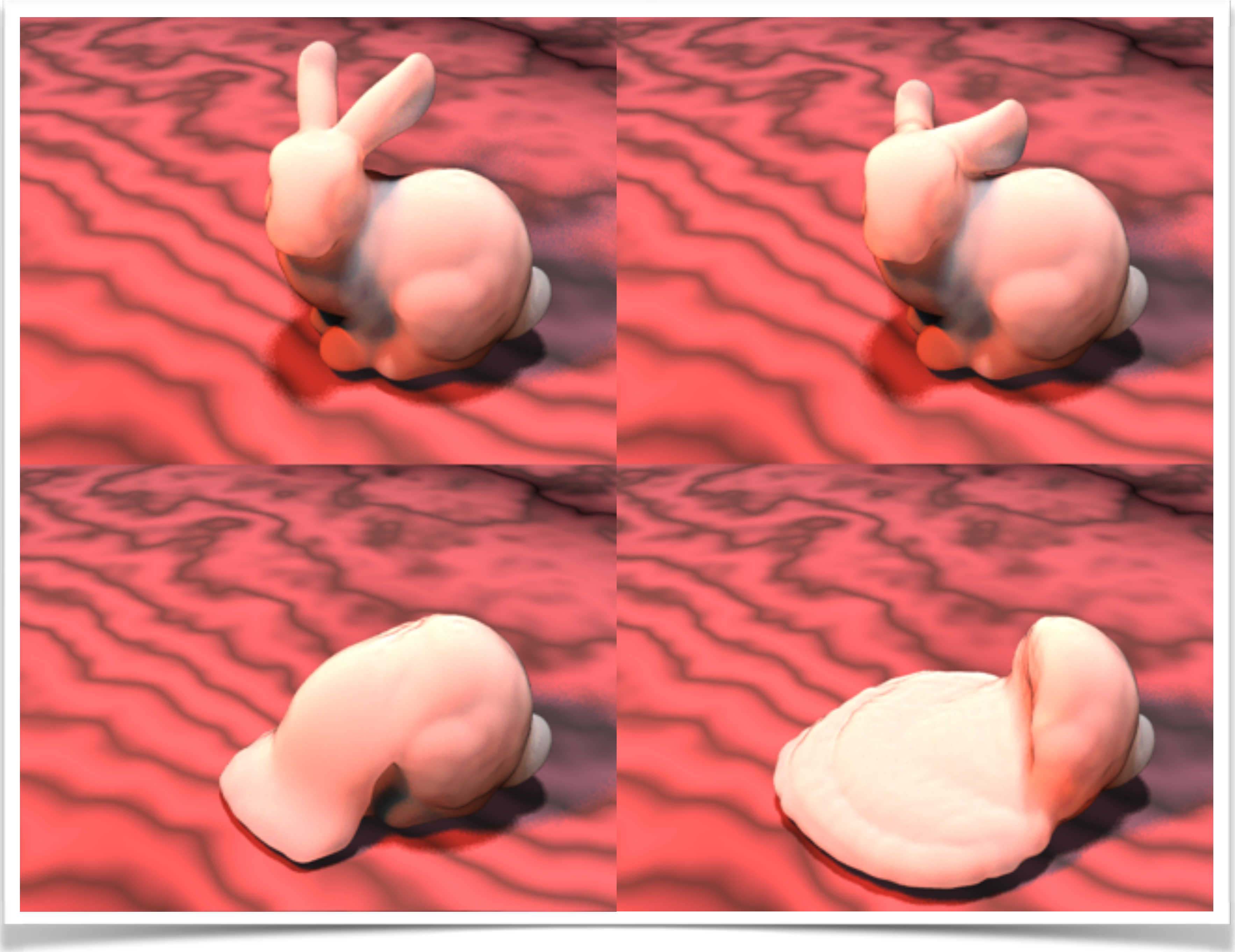
- Often ignored: advection causes enough diffusion already



- For high-viscosity fluids, just use implicit integration

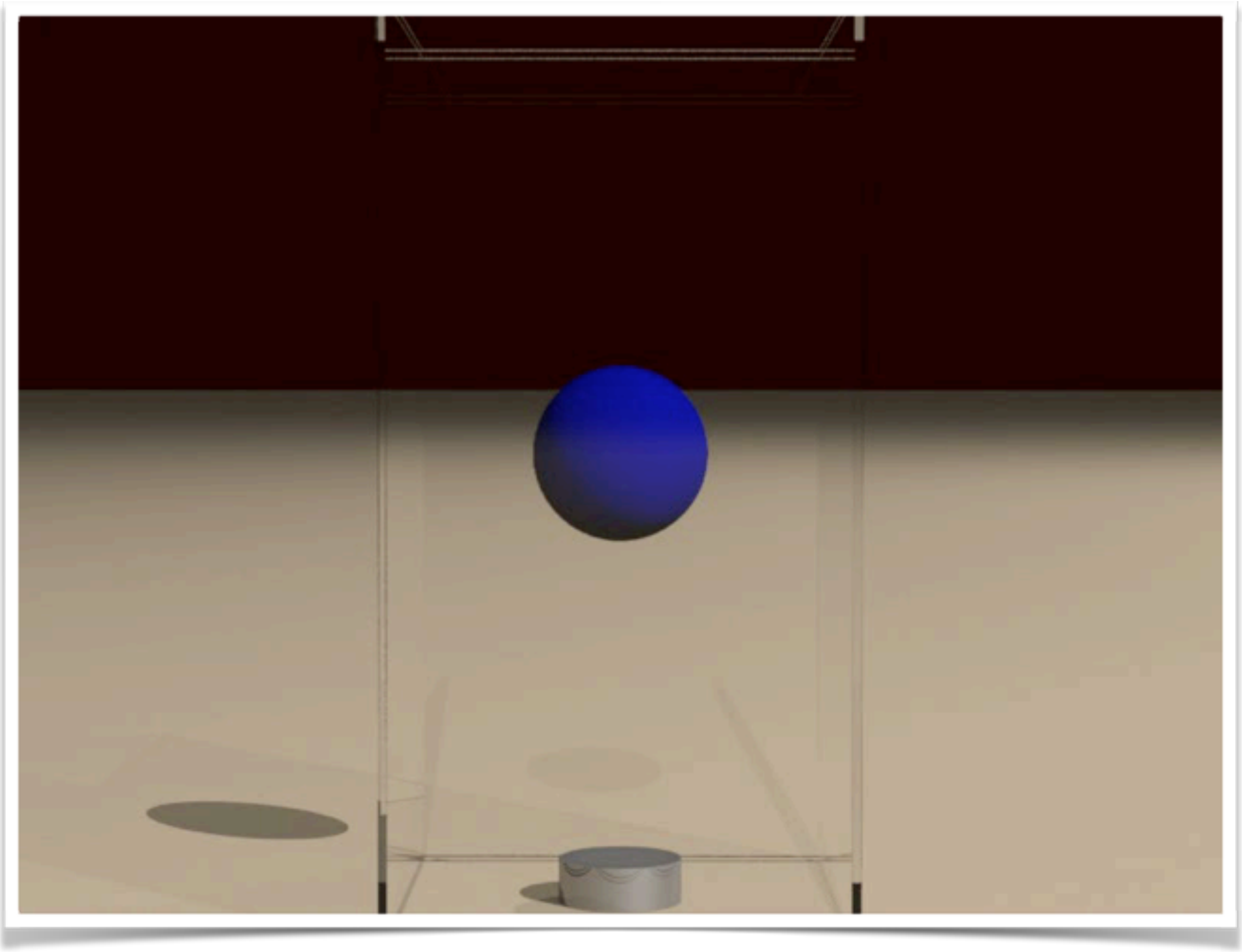
$$\mathbf{u}^{\text{new}} = \mathbf{u}^{\text{old}} + \frac{\mu}{\rho} \nabla^2 \mathbf{u}^{\text{new}} \Delta t$$

Viscosity



Carlson, Mucha, Van Horn, and Turk, 2002

Smoke simulation



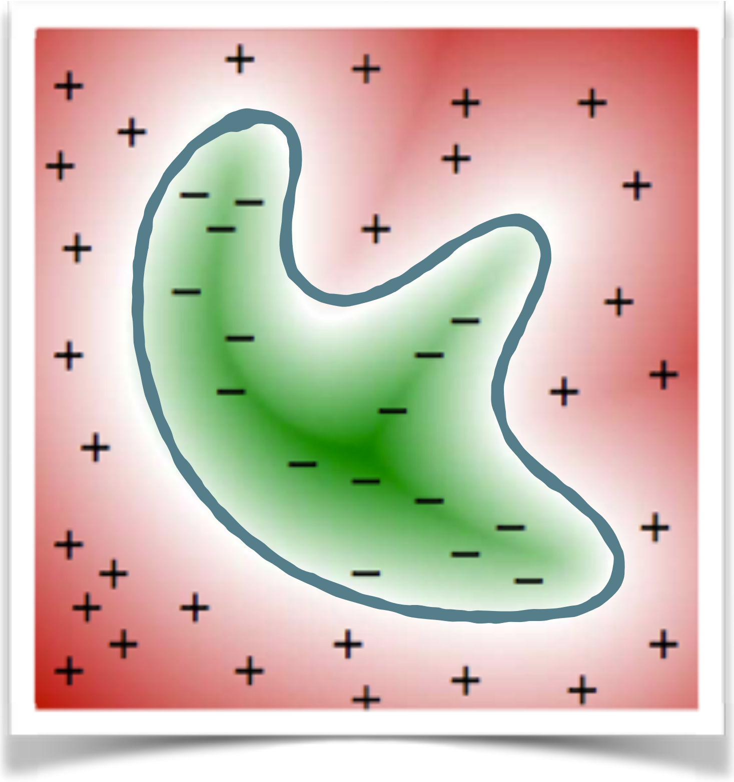
Lentine, Zheng, and Fedkiw, 2010

Surface tracking

- How to represent the surface of a liquid?

Option 1: Level set method

- Store the **signed distance** to surface $\phi(\mathbf{x})$ on grid cells
  - Advect forward at each time step
  - Surface is the level set (isosurface) at $\phi = 0$

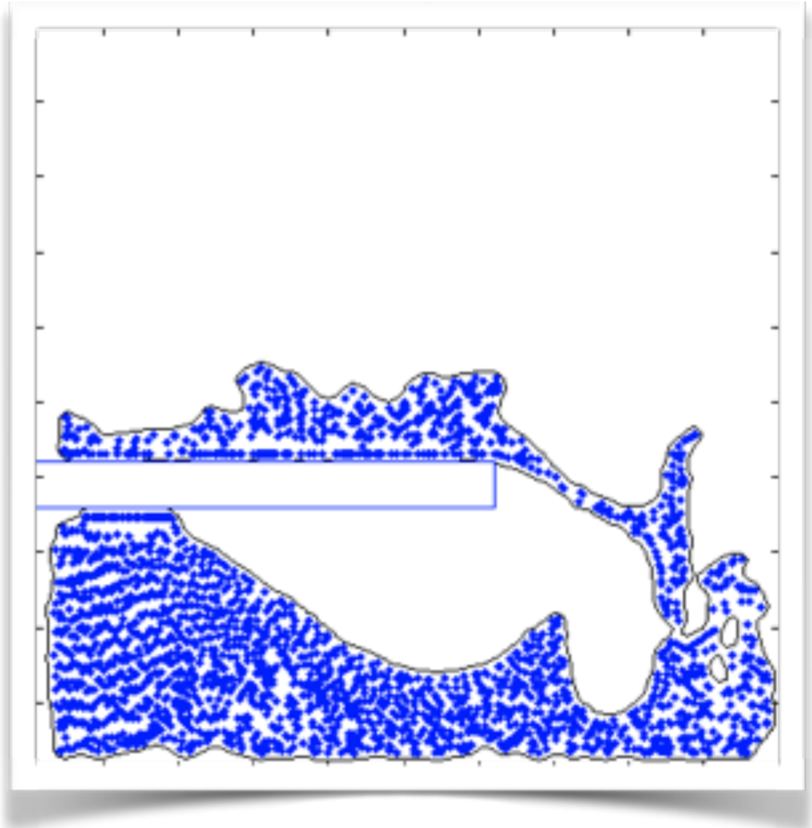


Surface tracking

- How to represent the surface of a liquid?

Option 2: Particles (easier)

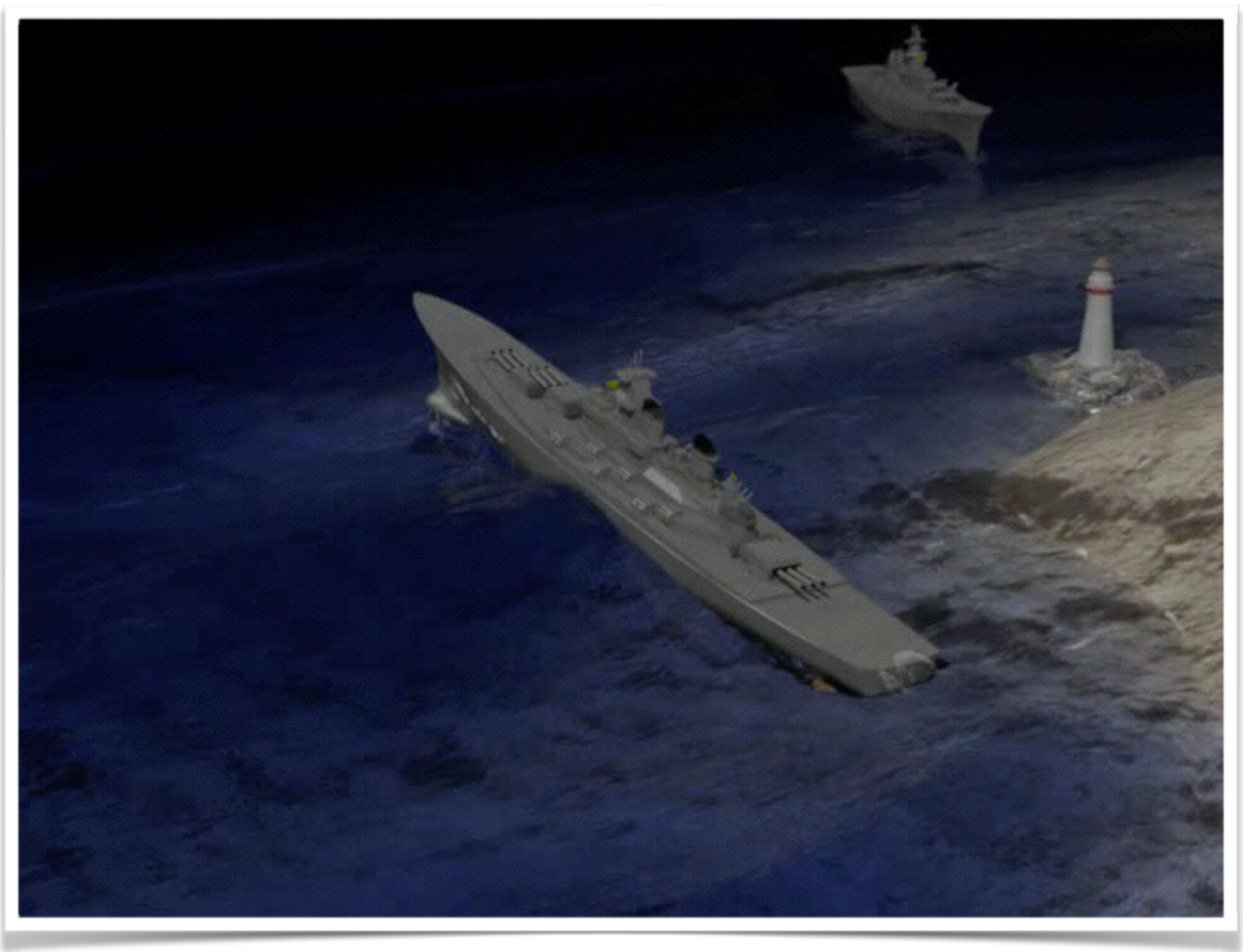
- Keep lots of particles in the fluid, passively advected with the flow
- Reconstruct surface as in SPH



Zhu and Bridson, 2005

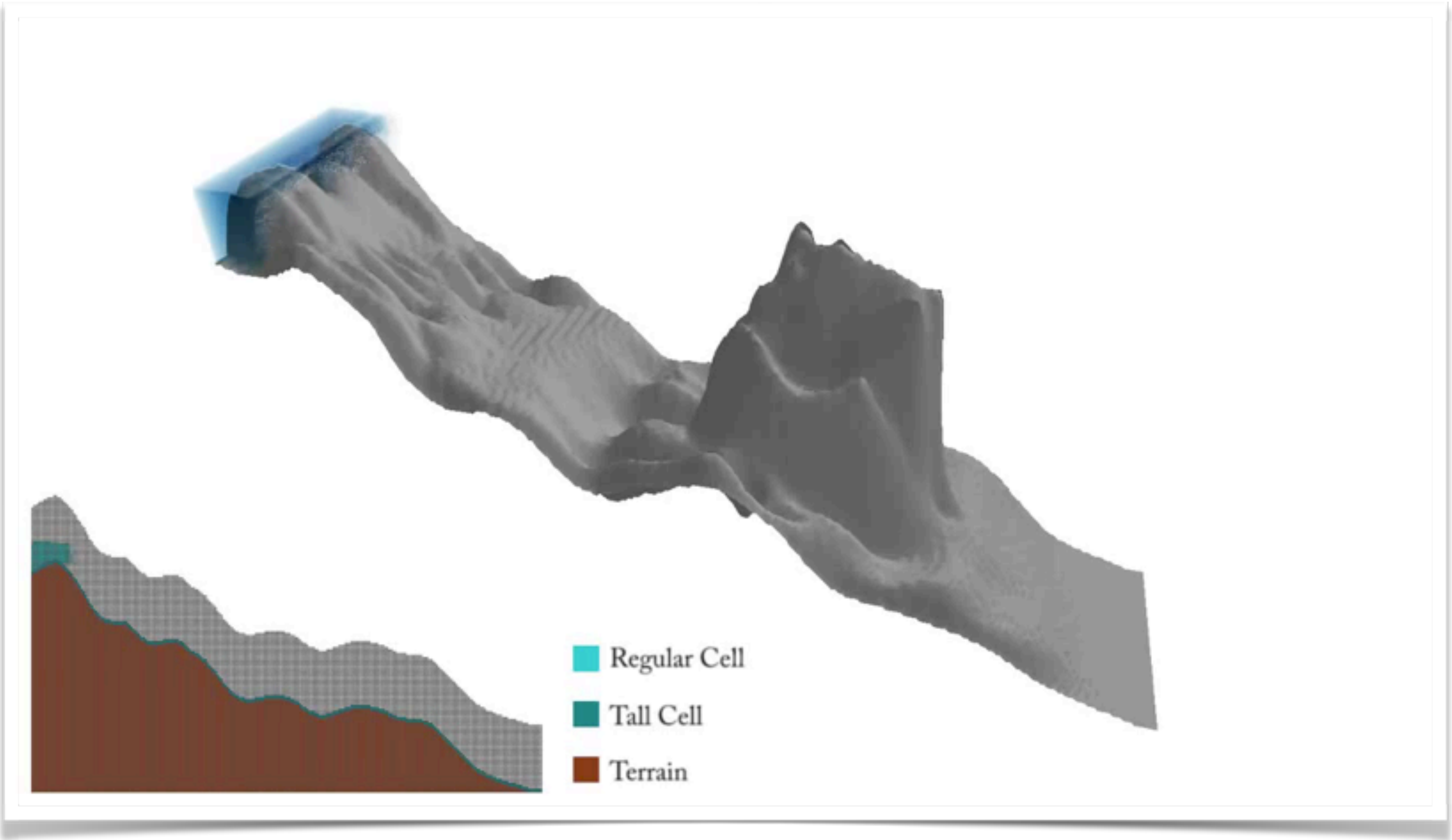
More options: level set + particles, volume-of-fluid, meshes, …

Liquid simulation



English, Qiu, Yu, and Fedkiw, 2013

Real-time liquid simulation



Chentanez and Müller, 2011

References

- Bridson and Müller-Fischer, "**Fluid Simulation for Computer Animation**", SIGGRAPH 2007 course notes
- Stam, "**Stable Fluids**", 1999
- Enright, Marschner, and Fedkiw, "**Animation and Rendering of Complex Water Surfaces**", 2002
- Zhu and Bridson, "**Animating Sand as a Fluid**", 2005 (for particle-based surface tracking, and more)