CS 170 Fall 2006 — Discussion Handout #10

December 10, 2006

1. **Local Search for Minimum Spanning Trees**
   Consider the set of all spanning trees (not just minimum ones) of a weighted, connected, undirected graph $G = (V, E)$. Recall that adding an edge e to a spanning tree T creates an unique cycle, and subsequently removing any other edge $e' \neq e$ from this cycle gives back a different spanning tree T'. We will say that T and T' differ by a single edge swap (e, e') and that they are neighbors.

   (a) Show that it is possible to move from any spanning tree T to any other spanning tree T' by performing a series of edge-swaps, that is, by moving from neighbor to neighbor. At most how many edge-swaps are needed?

   (b) Show that if T' is an MST, then it is possible to choose these swaps so that the costs of the spanning trees encountered along the way are non-increasing. In other words, if the sequence of spanning trees encountered is $T = T_0 \to T_1 \to T_2 \to T_k = T'$; then $\text{cost}(T_{i+1}) \leq \text{cost}(T_i)$ for all $i < k$.

   (c) Consider the following local search algorithm which is given as input an undirected graph G.
   `Let T be any spanning tree of G`
   `while there is an edge-swap` (e, e') `which reduces cost(T):`
   $T \leftarrow T + e - e'$
   `return T`
   Show that this procedure always returns a minimum spanning tree. At most how many iterations does it take?

2. **Miminum Steiner Tree**
   In the MINIMUM STEINER TREE problem, the input consists of: a complete graph $G = (V, E)$ with distances d_{uv} between all pairs of nodes; and a distinguished set of terminal nodes $V' \subset V$. The goal is to find a minimum-cost tree that includes the vertices V'. This tree may or may not include nodes in $V - V'$.

   Suppose the distances in the input are a metric. Show that an efficient ratio-2 approximation algorithm for MINIMUM STEINER TREE can be obtained by ignoring the nonterminal nodes and simply returning the minimum spanning tree on V'. (Hint: Recall our approximation algorithm for the TSP.)

3. **Approximation for Max Cut**
   In the MAXIMUM CUT problem we are given an undirected graph $G = (V, E)$ with a weight $w(e)$ on each edge, and we wish to separate the vertices into two sets S and $V - S$ so that the total weight of the edges between the two sets is as large as possible. For each $S \subseteq V$ define $w(S)$ to be the sum of all $w(e)$ over all edges $\{u, v\}$ such that $|S \cap \{u, v\}| = 1$. Obviously, MAX CUT is about maximizing $w(S)$ over all subsets of V . Consider the following local search algorithm for MAX CUT:
   `start with any` $S \subseteq V$
   `while there is a subset` $S' \subseteq V$ `such that` $||S| - |S'|| = 1$, $|S' \cap S| = |S| - 1$ `and` $w(S') > w(S)$ `do:`

`set` $S = S'$

(a) Show that this is an approximation algorithm for MAX CUT with ratio 2.

(b) But is it a polynomial-time algorithm?

4. **Quantum Measurements**

   (a) What is the probability of measuring $|0\rangle$ in the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$?

   (b) Consider the state $|\Phi\rangle = \frac{1}{\sqrt{3}}|0\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle \otimes (|10\rangle)$. We now measure the first qubit. What is the probability of measuring $|0\rangle$ for the first qubit? If we measure a $|0\rangle$ for the first qubit, what is the state of the system after the measurement?