Mathematical Aspects/Derivation of Histogram Equalization

- Consider continuous values of intensity rather than discretized.
- r = gray level of image to be equalized

  r normalized to $[0, 1]$

  $0 \rightarrow$ dark = Black

  $1 \rightarrow$ Bright = White.

- Goal: Design a Transformation.

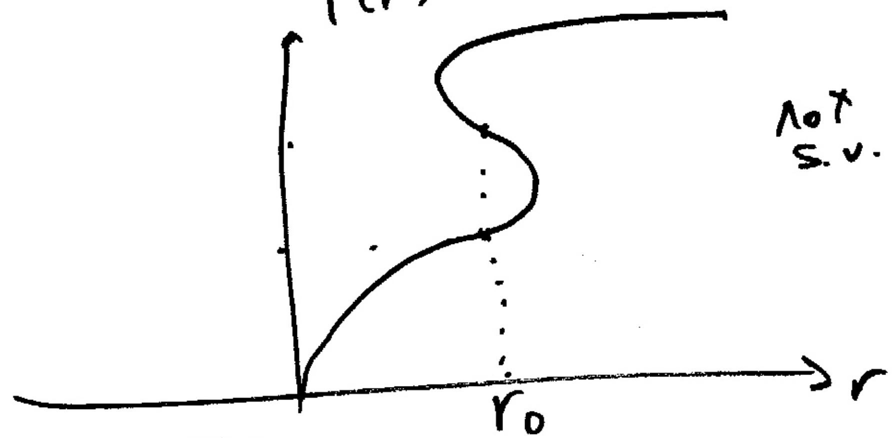
histogram matching or equalization $\longleftarrow T(r) = S$ $\quad 0 \leq r \leq 1$

$\quad 0 \leq S \leq 1$

- Assumptions about $T(r)$.

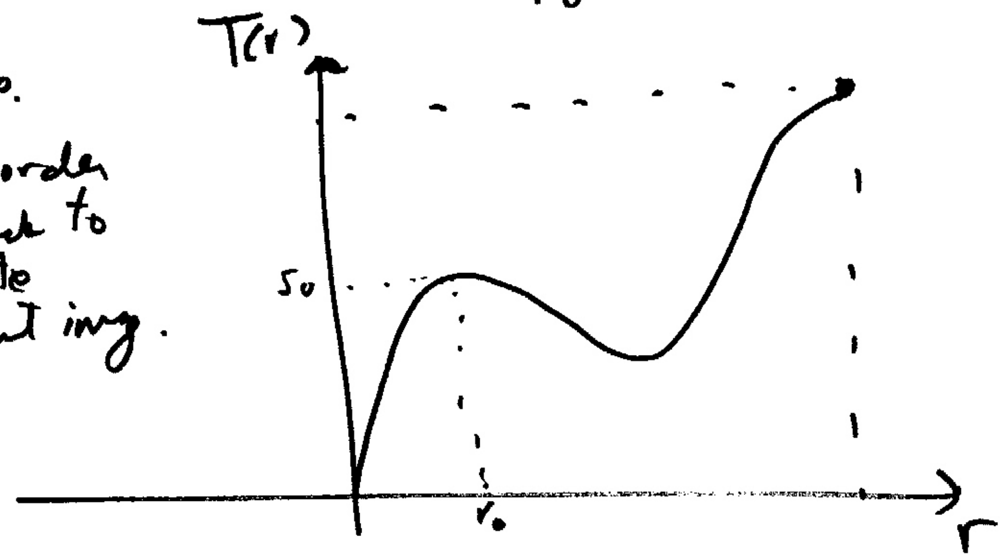
(a) $T(r)$ single valued and monotically increasing in the interval $0 \le r \le 1$

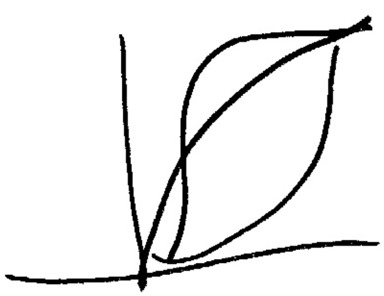
→ for inverse Transformation to exist



not s.v.

→ preserve increasing order from black to the white in output img.





(b) $0 \leq T(r) \leq 1$ for $0 \leq r \leq 1$

output grey levels in the same range as input levels.

$$T(r) = s$$

$$T^{-1}(s) = r$$

r = input intensity
s = output intensity.

<u>Result from prob Theory</u>:

If $T^{-1}(s)$ satisfies condition (a) Then.

$$p_s(s) = p_r(r) \left| \frac{dr}{ds} \right|$$

Consider CDF: as a Transformation.

$$S = T(r) = \int_0^r P_r(\omega)\,d\omega$$

$$\frac{ds}{dr} = \frac{d}{dr}\left\{\int_0^r P_r(\omega)\,d\omega\right\} = P_r(r)$$

$$P_s(s) = P_r(r)\left|\frac{1}{P_r(r)}\right| = 1 \qquad 0 \le s \le 1$$

Words: If $T(r)$ is just a CDF or just the integral of input pdf $(P_r(r))$ then ~~and~~ applying $T(r)$ results in a image whose pdf $(P_s(s))$ is Uniform

Discrete Case

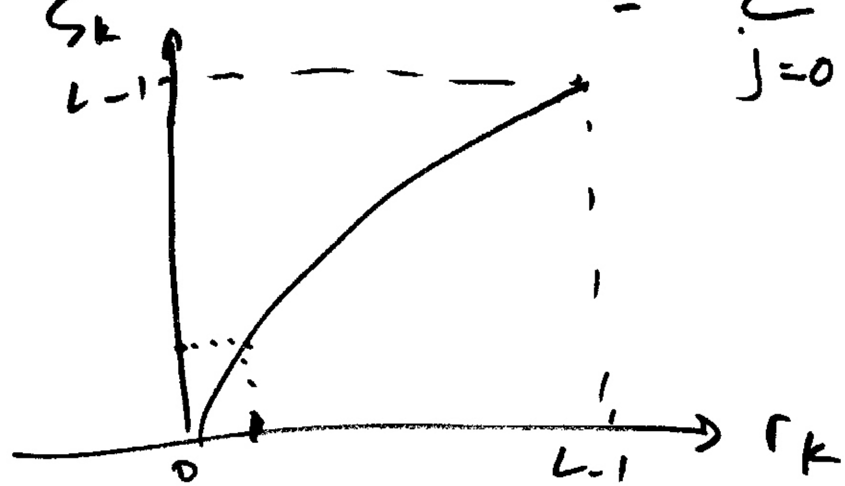
r_k = discrete intensity value. $\quad k = 0, \ldots, L-1$

$$p_r(r_k) = \frac{n_k}{n} \qquad k = 0, \ldots, L-1$$

n_k = # of pixels that have intensity r_k.

$$s_k = T(r_k) = \sum_{j=0}^{k} p_r(r_j)$$

$$= \sum_{j=0}^{k} \frac{n_j}{n} \qquad k = 0, \ldots, L-1$$



Histogram Matching

$T(r) = s$

Rather than $p_s(s)$ uniform, we want $p_s(s)$ to match a "desired" pdf
given

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r = pixel values before matching
z = pixel " after "

Can compute $p_r(r)$ from given image.
Know, given $p_z(z)$

<u>Goal</u>: what is the transform $r \longrightarrow z$?

Approach:

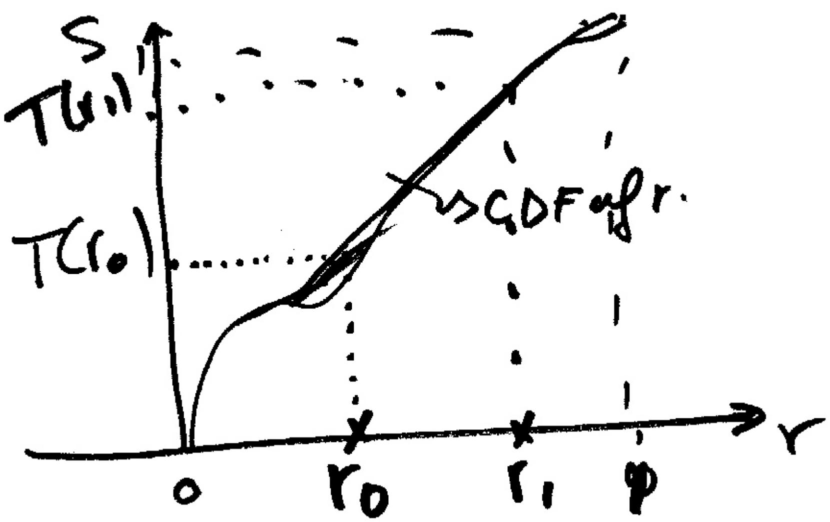
$$S = T(r) = \int_0^r P_r(w)\,dw \longrightarrow \text{CDF of } r.$$

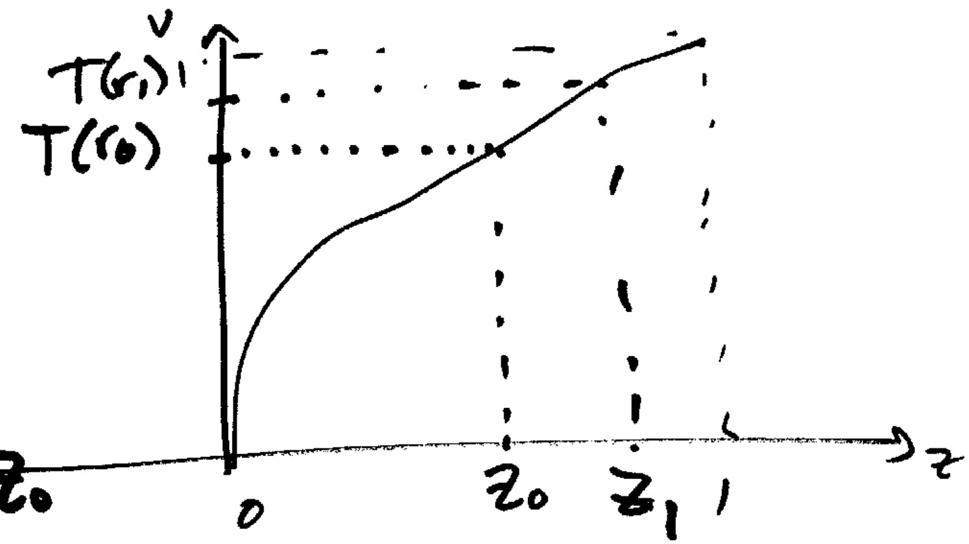
$$v = G(z) = \int_0^z P_z(t)\,dt \longleftarrow \text{CDF of } z.$$

Histogram matching

$G(z) = T(r)$

$z = G^{-1}\{T(r)\}$





$r_0 \longrightarrow$ how to find z_0

Lookuptable

r_0	z_0
r_1	z_1

$\rightarrow$ Matched histogram of $P_z(z)$ desired

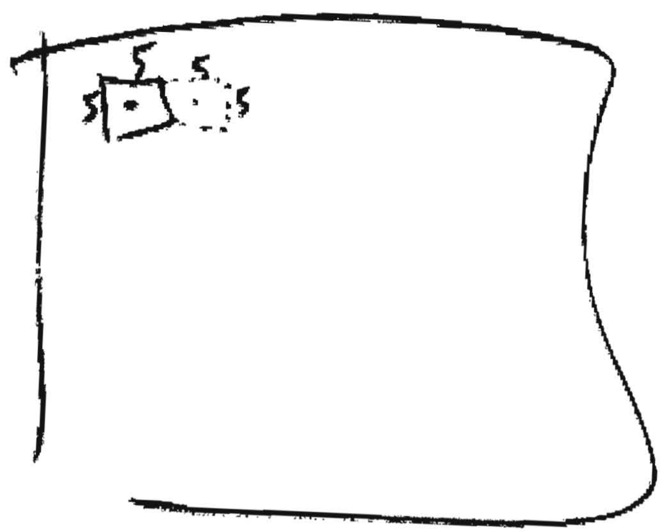
Local Enhancements

Using local histogram Analysis

Define square or rectangle neighborhood around each pixel, move center of "window" across the image.

At ~~each~~ each location : compute histogram in the window

Do hist. eq.



Use local stats for Enhancement

stat $\longrightarrow$ mean, variance:

global mean $= M_G = \sum_{i=0}^{L-1} r_i \, p(r_i)$

$r_i \rightarrow$ intensity level i

$i = 0, \ldots L-1$

global variance $= Var_G = \sum_{i=0}^{L-1} (r_i - M_G)^2 \, p(r_i)$

Local mean:

Let S_{xy} neighborhood, subimage centered around (x,y).

local mean $= m_{S_{xy}} = \sum_{(s,t) \in S_{xy}} r_{st} \, p(r_{st})$

local variance $= \sigma^2_{S_{xy}} = \sum_{(s,t) \in S_{xy}} [r_{st} - m_{S_{xy}}]^2 \, p(r_{st})$

Goal : Enhance dark areas while leaving the bright area as unchanged as possible.

- Detect region that have both.

(a) dark → local mean has to be small compared to global mean.

$$m_{S_{xy}} \le k_0 \, m_G$$

local mean

$0 \le k_0 \le 1$ constant

global mean

$k_0 \simeq 0.4$

(b) Low contrast

$$\sigma_{S_{xy}} \le k_2 \, \sigma_G$$

local var

$k_2 > 0$ constant

global var

$k_2 \simeq 0.4$

(c) not too low of contrast.
leave flat regions unchanged

$$\sigma_{S_{xy}} \geq K_1 \, Var_G$$

$K_1 \simeq 0.01$

$$g(x,y) = \text{processed} = \begin{cases} E\, f(x,y) & \text{if } M_{S_{xy}} \leq K_0 M_G \\ & K_1 Var_G \leq \sigma_{S_{xy}} \leq K_2 Var_G \\ 0 & \text{otherwise.} \end{cases}$$

Enhancement using Arithmetic / Logic Operations

Logic: AND . OR NOT $\longrightarrow$ functionally complete.

NAND $\longrightarrow$ functionally complete.

NOT $\longrightarrow$ negative transformation

AND OR $\longrightarrow$ Masking.

Image Subtraction

$$g(x, y) = f(x, y) - h(x, y)$$

mask.