Reconstruction of 2-D Signals from Partial Fourier Information

- motivation: electron microscopy.
  optical astronomy
  - crystallography . . .

- 3 problems:
  1. FTM = magnitude of Fourier Transform
  2. Recon from Phase Fourier Transform
  3. Recon from level crossings

Recon from FTM

1_D case:

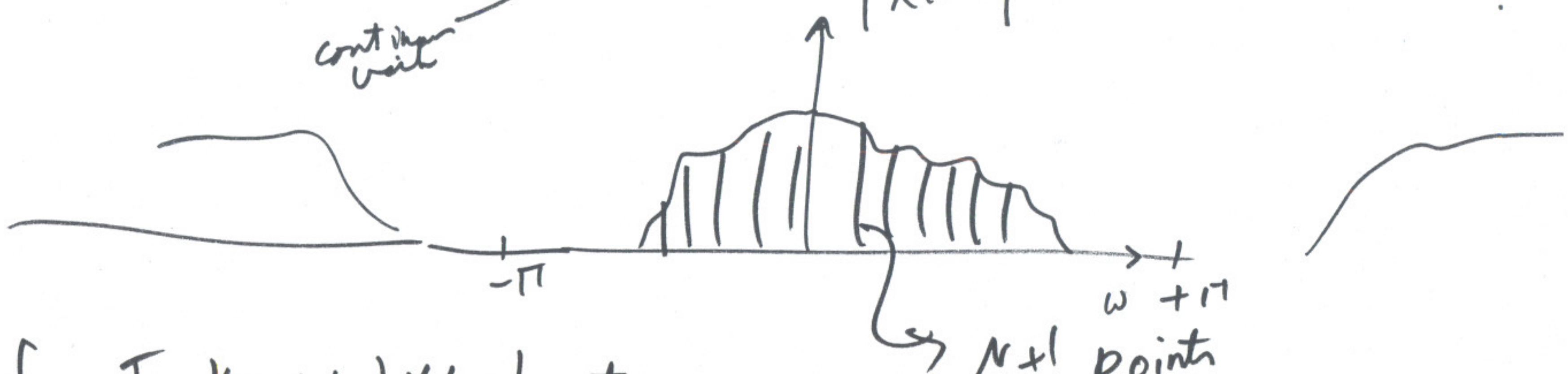
- $x(n)$: discrete time - 1-D -
  $N+1$ nonzero points.
  $n=0, \ldots, N$

$e^{-j\omega} = y$

$$D.T.F.T\{x(n)\} = X(\omega) = \sum_{n=0}^{N} x(n) e^{-j\omega n}$$

If We know $|X(\omega)| \longrightarrow$ Can we get $x(n)$?

continuous variable



If I know $|X(\omega)|$ at $N+1$ pts $\Longrightarrow$ cannot recon signal

$$
\begin{bmatrix}
e^{-0\cdot j\omega_0} & e^{-j1\omega_0} & e^{-j2\cdot\omega_0} & \cdots \\
e^{-0\cdot j\omega_1} & e^{-j1\cdot\omega_1} & e^{-j2\omega_1} & \cdots e^{-j2\omega_1} \\
\vdots & & & \\
e^{-0\cdot j\omega_N} & e^{-j\omega_N} & e^{-j2\omega_N} & \cdots e^{-jN\omega_N}
\end{bmatrix}
\begin{bmatrix}
x(0) \\ x(1) \\ \vdots \\ x(N)
\end{bmatrix}
=
\begin{bmatrix}
|X(\omega_0)|\phi \\ |X(\omega_1)|\phi \\ \vdots \\ |X(\omega_N)|\phi
\end{bmatrix}
$$

$$\sum_{n=0}^{N} x(n)\, y^{n} = X(\omega)$$

Sampling a 2-D poly of degree at arbitrary N points results in unique recon of polynomial

$$
\underset{4N}{\begin{bmatrix} e^{-0jw_0} & e^{-j1w_0} & \cdots \\ e^{-0jw_1} & \cdots & \\ \vdots & & \ddots \\ e^{-0jw_N} & e^{-j1w_N} & \cdots \end{bmatrix}} \underset{N}{\begin{bmatrix} x(0) \\ \vdots \\ x(N) \end{bmatrix}} = \underset{4N}{\begin{bmatrix} |X(w_0)|^2 \\ |X(w_1)|^2 \\ \vdots \\ |X(w_N)|^2 \end{bmatrix}}
$$

Q: Can we uniquely reconstruct if we oversampled F.D.?

A for 1D signal $\longrightarrow$ No 2^N ambiguity

MD signals $\longrightarrow$ yes

1-D

auto-correlation fn of $x(n)$.

$$r(n) = \sum_{l} x(l)\, x^*(l+n)$$

$$R(\omega) = \text{D.T.F.T}\{r(n)\} = \sum r(n) e^{-j\omega n} = |X(\omega)|^2 =$$

$$\Longrightarrow F^{-1}\{|X(\omega)|^2\} = r(n)$$

Q: Can we obtain $x(n)$ from $r(n)$?

$$\text{Z.T.}\{r(n)\} = R(z) = X(z)\, X^*\left(\frac{1}{z^*}\right) = \sum_{n=-N}^{+N} r(n) z^{-n}$$

Associated polynomial to $R(z) \triangleq P_r(z) = \sum_{n=0}^{2N} r[N-n] z^n$

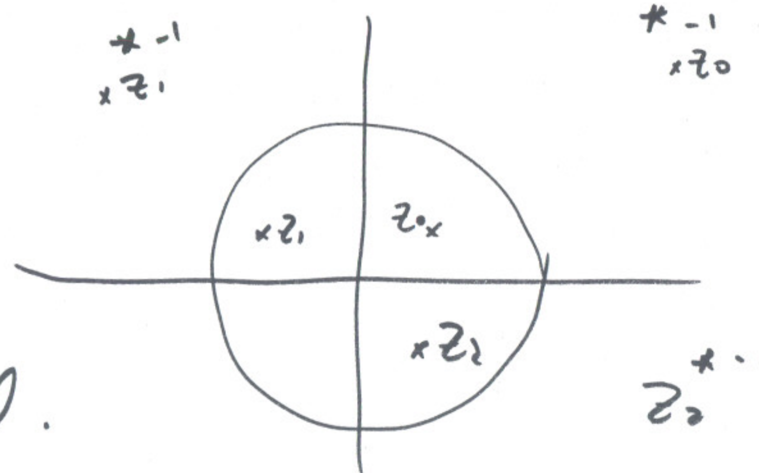
symetric

Since coeff $P_r(z)$ are symetric $\Longrightarrow$

If z_0 is a zero of $P_r(z)$, so is z_0^{*-1}

$\Rightarrow$ Can factor $P_r(z)$

$$P_r(z) = A \prod_{i=1}^{N} (z - z_i)(1 - z_i^* z)$$



Def mirror of a polynomial.

Associated with any polynomial, there is a mirror polynomial consisting of coefficients in reverse order and conjugated

If $P(z) = \sum_{n=0}^{N} p_n z^n \xrightarrow{\text{mirror}} \hat{p}(z) = \sum_{n=0}^{N} p_{N-n}^* z^n$

- Assume $y(n) \longrightarrow$ autocorrelation $r(n)$

$$P_r(z) = P_y(z)\, \hat{P}_y(z)$$

2^N ways in order to generate $P_y(z)$.

$$P_y(z) = \sqrt{A} \prod_{i \in I} (z - z_i) \prod_{j \notin I} (z - z\, z_j^*)$$

I any subset of $[1, \ldots, N]$

(z_0, z_0^{*-1}) $\quad P(z)$

(z_1, z_1^{*-1}) $\quad P_r(z)$

$\vdots$

$r(n)$

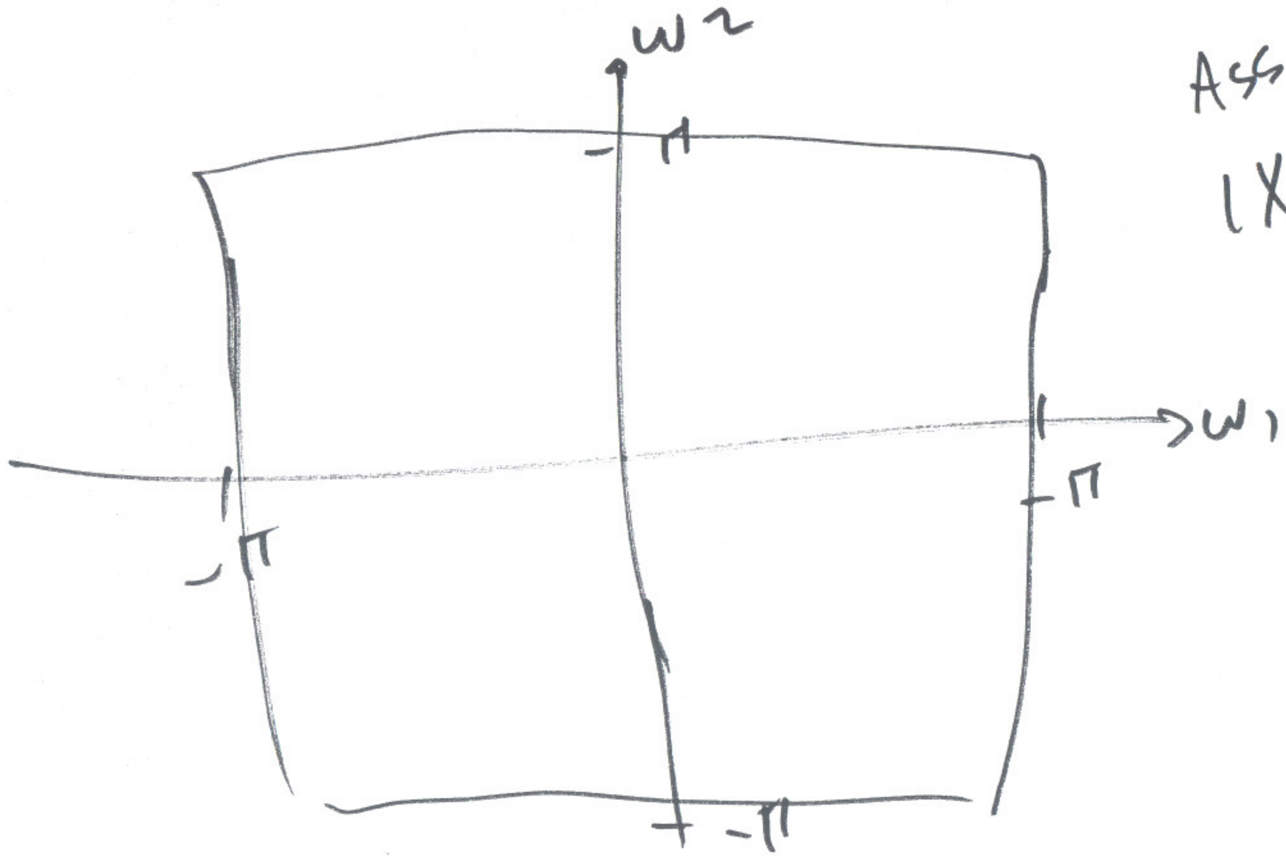
$\downarrow$

$x(n)$

(z_N, z_N^{*-1})

$2^N \; x(n) \longrightarrow$ same $r(n)$

2-D case FTM



Assume $|X(\omega_1, \omega_2)|^2$

$x(n_1, n_2)$: $N \times N$

$\Rightarrow$ in 2-D, unlike 1-D, most 2D polynomials are irreducible $\Longrightarrow$ non factorable.

1982, M. Hayes : If $x(n_1, n_2)$ has irreducible associated polynomials, then all other $y(n)$ which have same F.T.M are equivalent to x. Equivalent means:

$$y(n_1, n_2) \sim x(n_1, n_2) \quad \text{if}$$

$$\cancel{y(n_1, n_2) = e^{j\theta} x(n_1, n_2)}$$

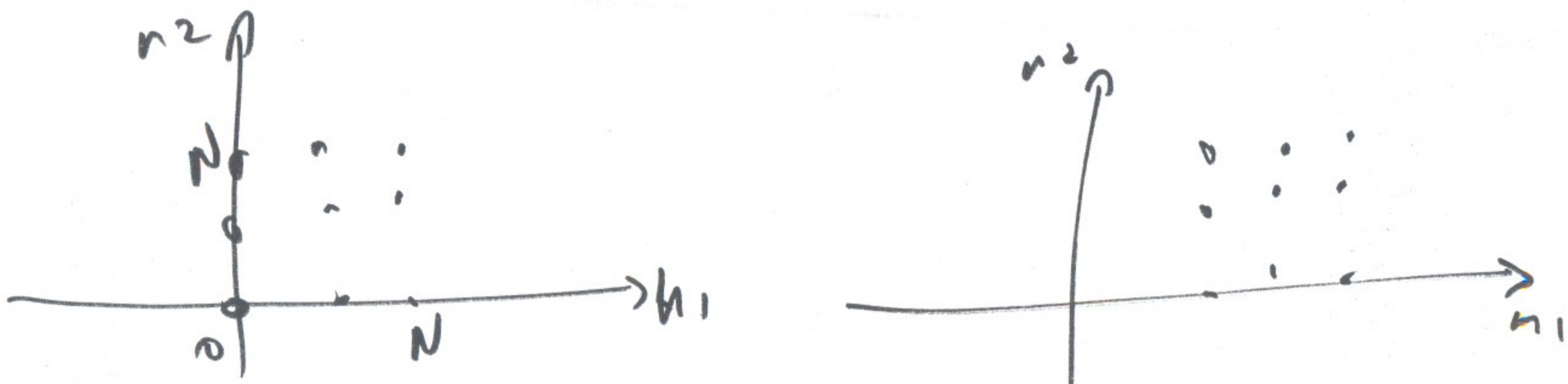
$$y(n_1, n_2) = e^{j\theta} x(k_1 \pm n_1, k_2 \pm n_2)$$

- observation ① : $x(n_1, n_2) \longrightarrow |X(\omega_1, \omega_2)|^2$

$$e^{j\theta} x(n_1, n_2) \longrightarrow |X(\omega_1, \omega_2)|^2$$

- observation ② $x(n_1, n_2) \longrightarrow |X(\omega_1, \omega_2)|^2$

shift by k_1, k_2 in space domain $\Longrightarrow$ same FTM



- Observation 3: Rotatiation factor.

$$x(n_1, n_2) \longrightarrow |X(\omega_1, \omega_2)|^2$$

$$y(n_1, n_2) = x(N - n_1, N - n_2)$$

also has same FTM.

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1) Assume signal $x(n_1, n_2)$ is real

$$e^{j\theta}, \quad \theta = 0, \text{ or } \theta = \pi$$

- Assume signal is positive $\Longrightarrow \theta = 0$

extent of $x(n_1,n_2)$ is known.

② - Assumption: irreducible.

③ Z.T of signal $x \Longrightarrow$ Nail signal to factor of 2:

Hayes. $\Longrightarrow$ either $x(n_1,n_2)$ or $x(N-n_1, N-n_2)$

same FTM.

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① assume x Real, ~~we~~ ~~ken~~ and positive.

Know extent.

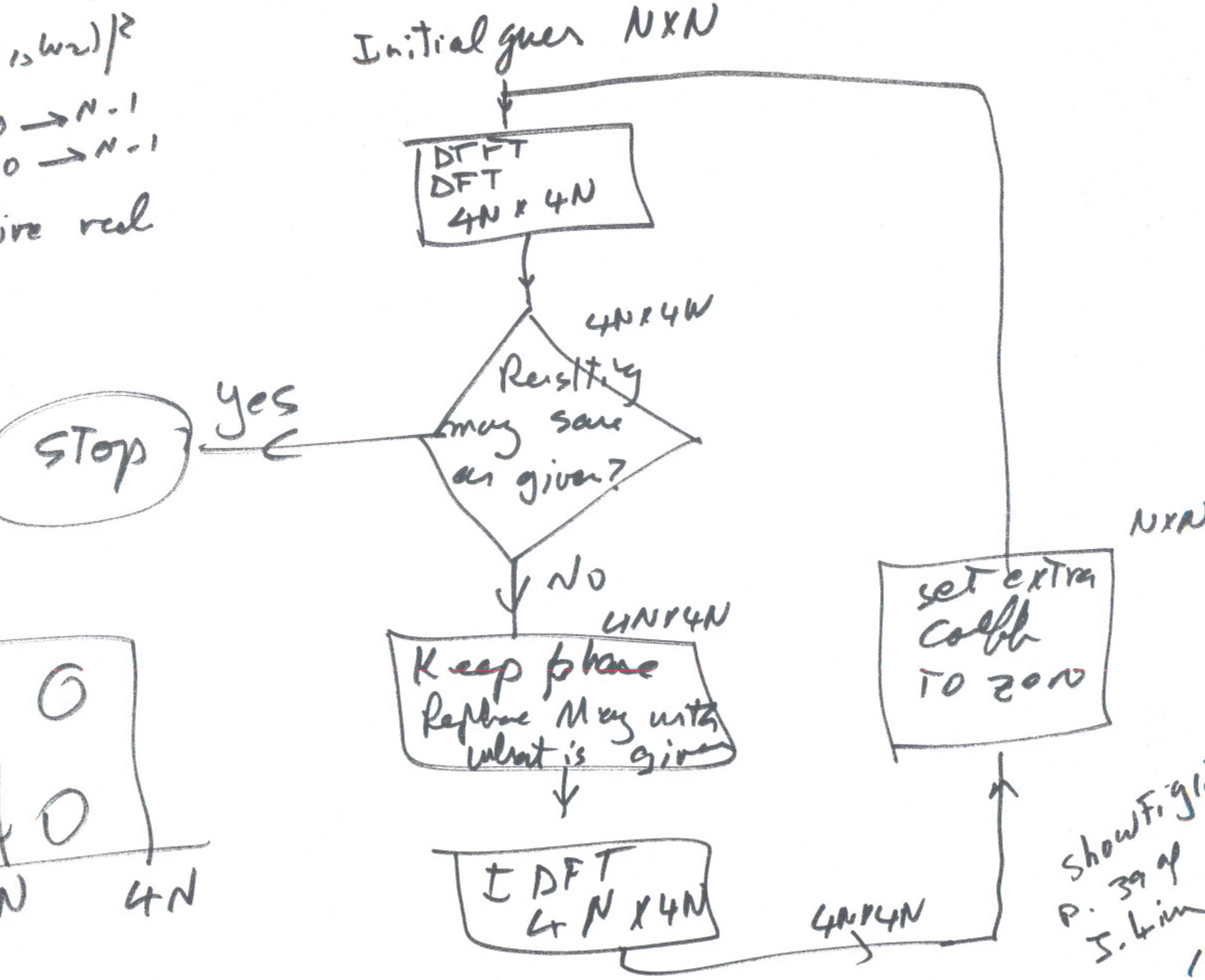
FTM can be used to recover either x or its reflection "uniquely"

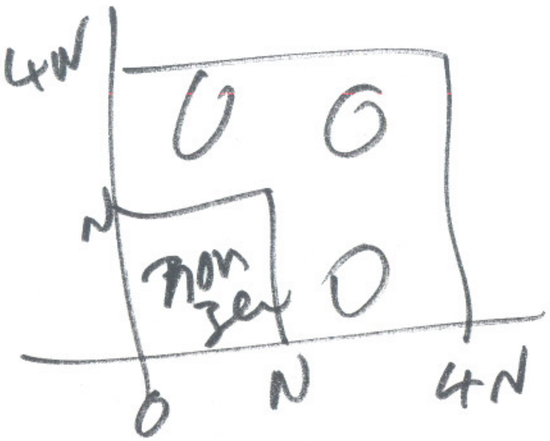
proposed iterative Alg

Assume known
$4N \times 4N$ samples
of $|X(\omega_1, \omega_2)|^2$

$N \times N$ $\quad n_1 : 0 \rightarrow N-1$
$\quad\quad\quad n_2 : 0 \rightarrow N-1$

x is positive real





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