MUlTi_Resolution Expansion

- Scaling fn ϕ : creates a series of approximations of a fn each differing by a factor of 2 in resolution

- Function ψ : (wavelet) encodes diff between adjacent approximations.

Series Expansion

Expand fn $f(x)$ as:

$$f(x) = \sum_{k} \alpha_k \, \phi_k(x)$$

$\phi_k(x) \triangleq$ real valued expansion functions

$\alpha_k \triangleq$ " " " coefficients

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If expansion unique i.e. only one set of α_k for $f(x)$

$\Longrightarrow$ ϕ_k = basis functions. $\{\phi_k\}$ = basis for class of fns.

function space: $V \triangleq \overline{\operatorname{Span}_k \{\phi_k(x)\}}$ closed span of expansion set

$f(x) \in V \Longrightarrow f(x)$ is in closed span of $\{\phi_k(x)\}$

and can be written as $f(x) = \sum_k \alpha_k \phi_k(x)$

Dual function $\{\tilde{\phi}_k(x)\}$ To $\{\phi_k(x)\}$

$$\alpha_k = \langle \tilde{\phi}_k(x), f(x) \rangle = \int \tilde{\phi}_k^*(x) f(x)\, dx$$

Consider 3 cases:

1. Expansion fns form an orthonormal basis for V:

$$\langle \phi_j(x), \phi_k(x) \rangle = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$\Longrightarrow$ $\phi_k(x) = \tilde{\phi}_k(x)$ basis & dual same.

$\Longrightarrow$ $\alpha_k = \langle \phi_k(x), f(x) \rangle$

2. Expansion fn orthogonal but not orthonormal

$$\langle \phi_j(x), \phi_k(x) \rangle = 0 \qquad j \neq k$$

$\Rightarrow$ basis fn and dual are bi-orthogonal

$$\alpha_k = \langle \tilde{\phi}_k(x), f(x) \rangle$$

$$\langle \phi_j(x), \tilde{\phi}_k(x) \rangle = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

3 More than one set of α_k in

$$f(x) = \sum_k \alpha_k \phi_k(x)$$

$\Rightarrow$ Exp fn and duals are "<u>Over complete</u>" or "<u>redundant</u>"

Form a <u>frame</u>

$$A \|f(x)\|^2 \leq \sum_k |\langle \phi_k(x), f(x) \rangle|^2 \leq B \|f(x)\|^2$$

for $A > 0$, $B < \infty$ $\forall f(x) \in V$

- If $A = B \longrightarrow$ Tight frame.

Daubechies 1992 $\quad f(x) = \frac{1}{A} \sum_k \langle \phi_k(x), f(x) \rangle \phi_k(x)$

Scaling functions

- Start with real, square integrable fn $\phi(x)$
- Build a set $\phi_{j,k}(x) = 2^{j/2}\,\phi(2^j x - k)$

$j, k \in \mathbb{Z}$ $\qquad \phi(x) \in L^2(\mathbb{R})$

- Denote subspace ~~span~~:

$$V_j = \operatorname{span}_k \{ \phi_{j,k}(x) \}$$

Then if $f(x) \in V_j \Rightarrow f(x) = \sum_k \alpha_k \, \phi_{j,k}(x)$

- Example Haar basis.

$$\phi(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Show Fig 7.11 G + W 3E

$f(x) \in V_0 \Rightarrow f(x) \in V_1 \quad : \quad V_0 \subset V_1$

For Haar

Multiresolution Analysis (Mallat)

(1) Scaling fn is $\perp$ to its integer Translates
(only for Haar)

(2)

$$\cdots\cdots \quad V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots\cdots$$

nesting of subspaces.

if $f(x) \in V_j$ Then $f(2x) \in V_{j+1}$

(3) Only fn common to all V_j is $f(x) \equiv 0$

$$V_{-\infty} = \{0\}$$

(4) Any fn can be represented with arbitray precision.

$$V_\infty = \{ L^2(R) \}.$$

- Can write $\phi_{j,k}$ as linear Combination of $\phi_{j+1,k}$

$$\phi_{j,k}(x) = \sum_n \alpha_n \phi_{j+1,n}(x)$$

$$\phi_{j,k}(x) = \sum_n h_\phi(n)\, 2^{\frac{j+1}{2}} \phi(2^{j+1} x - n)$$

Set $j = k = 0 \Rightarrow \phi_{0,0} = \phi(x)$

$$\phi(x) = \sum_n h_\phi(n) \sqrt{2}\, \phi(2x - n)$$

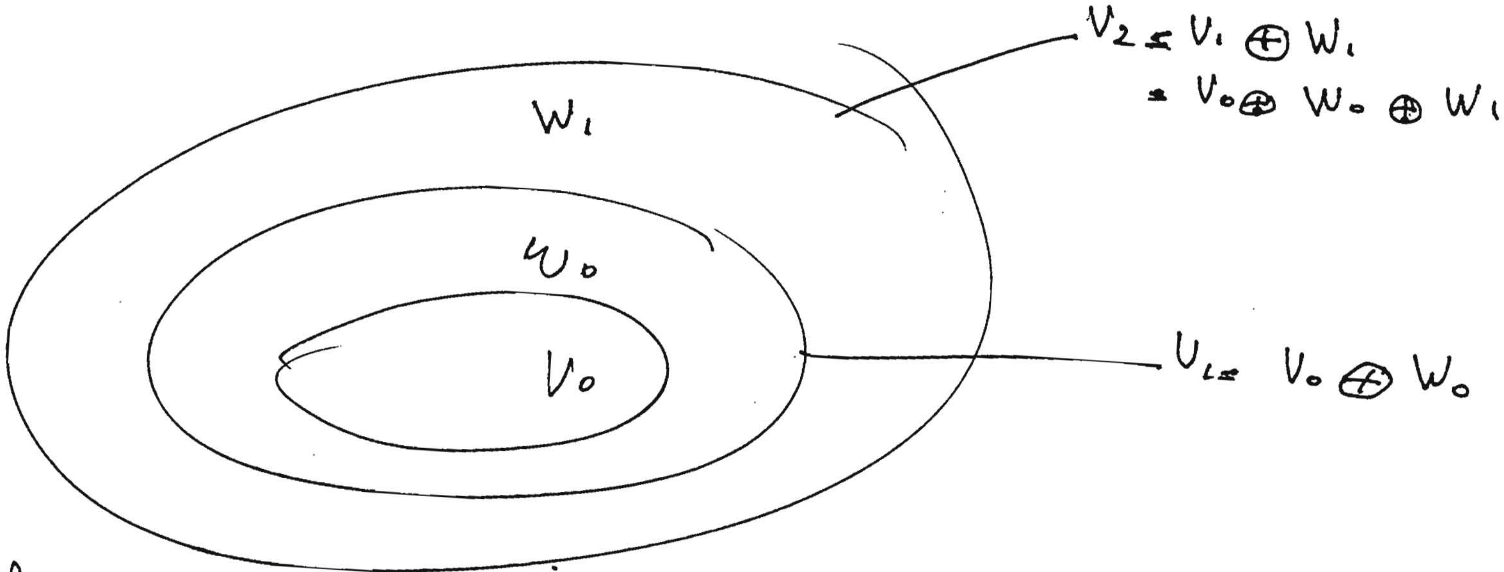
— $\phi(x)$ can be built from double resolution copies of itself. ie from $\phi(2x)$

— Expansion fn of V_j is linear comb of V_{j+1}

Fig 7.11f G+W

<u>Wavelet fns</u>

- Spans The difference between 2 adjacent subspace V_j and V_{j+1}



- $\psi_{j,k}(x) = 2^{j/2} \, \psi(2^j x - k) \qquad k \in \mathbb{Z}$
- $W_j = \underline{\text{Span}_k \{\psi_{j,k}(x)\}}$

$$V_{j+1} = V_j \oplus W_j$$

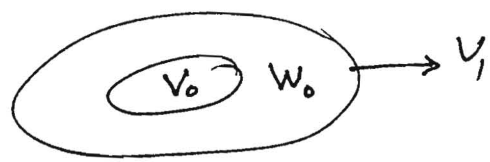
union of spaces

- Orthogonal Complement of V_j in V_{j+1} is W_j

$$\Longrightarrow \langle \phi_{j,k}, \psi_{j,l} \rangle = 0 \qquad \forall j,k,l \in \mathbb{Z}$$

- $$L^2(\mathbb{R}) = V_0 \oplus W_0 \oplus W_1 \oplus \cdots$$
$$= V_1 \oplus W_1 \oplus + W_2 \oplus \cdots$$
$$= \cdots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots$$

no need to deal with ϕ only ψ.

- 

If $f \in V_0$

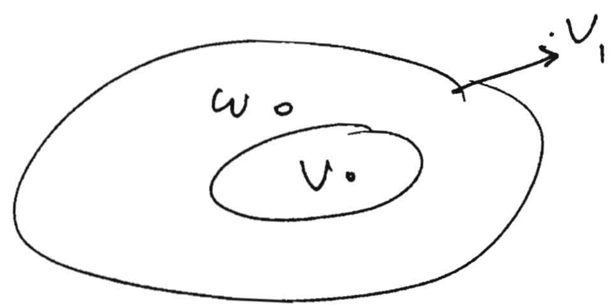
f is linear comb of scaling fn in V_0 + linear comb of wavelet from W_0

- $L^2(\mathbb{R}) = V_j \oplus W_j \oplus W_{j+1} \oplus \cdots\cdots$

j arbitrary.

$= V_0 \oplus W_0 \oplus W_1 \oplus W_2 + \cdots$

$= V_5 \oplus W_6 \oplus W_7 \oplus W_8 \oplus \cdots$



Basis for $V_0 \rightarrow \phi(x)$

" " $V_1 \rightarrow \phi(2x)$

" " $W_0 \rightarrow \psi(x)$

$\phi(x) \simeq$ linear combo of $\phi(2x)$

$\psi(x) \simeq$ " " " $\phi(2x)$

$$\psi(x) = \sum_n h_\psi(n) \sqrt{2}\, \phi(2x-n)$$

- <u>Theorem by Burrus</u> $\quad h_\psi(n) = (-1)^n h_\phi(1-n)$

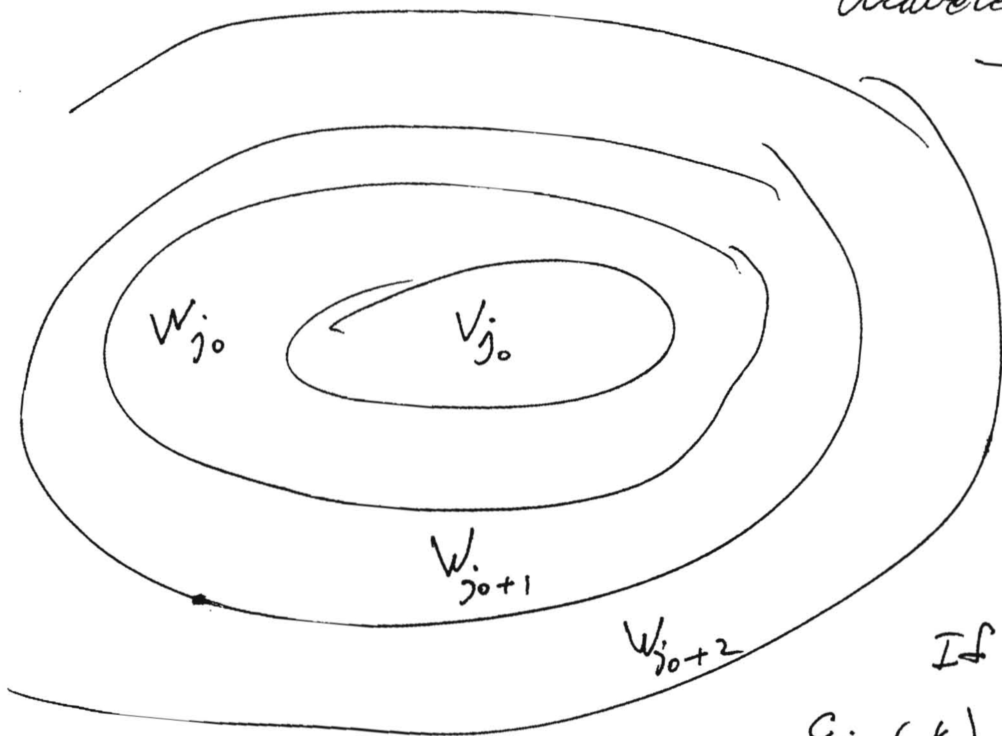
$\Rightarrow$ from $\phi(x) \rightarrow h_\phi \rightarrow h_\psi \rightarrow \psi$

$\Rightarrow$ For Haar

$$\psi(x) = \begin{cases} 1 & 0 \le x < 0.5 \\ -1 & 0.5 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Show Fig 7.14 G+W

<u>Wavelet Series Expansion.</u>



Arbitrary j_0.

$$f(x) = \sum_k c_{j_0}(k)\, \varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_k d_j(k)\, \psi_{j,k}(x)$$

If φ orthonormal or tight frame:

$$c_{j_0}(k) = \langle f(x), \varphi_{j_0,k}(x) \rangle = \int f(x)\, \varphi_{j_0,k}(x)\, dx$$

$$d_j(k) = \langle f(x), \psi_{j,k}(x) \rangle = \int f(x)\, \psi_{j,k}(x)\, dx$$

Show Fig 7.15

Discrete Wavelet Transform

So far ~~[illegible]~~ dealt with $f(x)$ x Real.

now deal with $f(n)$ n integer $\Rightarrow$ sequence not fn.

- Forward DWT coefficients for $f(n)$, (assuming Tight frame or orthogonal)

$$W_\varphi(j_0, k) = \frac{1}{\sqrt{M}} \sum_n f(n)\, \varphi_{j_0,k}(n)$$

sampled version of $\varphi_{j_0,k}(x)$

$$W_\psi(j, k) = \frac{1}{\sqrt{M}} \sum_n f(n)\, \psi_{j,k}(n) \qquad j \geq j_0$$

Then

$$f(n) = \frac{1}{\sqrt{M}} \sum_k W_\varphi(j_0, k)\, \varphi_{j_0,k}(n) \; +$$

$$\frac{1}{\sqrt{M}} \sum_{j=j_0}^{\infty} \sum_k W_\psi(j,k)\, \psi_{j,k}(n)$$

Continuous Wavelet Transform

- Already discussed last Time

$$W_\psi(s, \tau) = \int_{-\infty}^{+\infty} f(x)\, \psi_{s,\tau}(x)\, dx$$

$$\psi_{s,\tau}(x) = \frac{1}{\sqrt{s}}\, \psi\left(\frac{x-\tau}{s}\right)$$

s = scale τ = Translation

Inverse Wavelet

$$f(x) = \frac{1}{C_\psi} \int_0^{\infty} \int_{-\infty}^{+\infty} W_\psi(s,\tau)\, \frac{\psi_{s,\tau}(x)}{s^2}\, d\tau\, ds$$

where

$$C_\psi = \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\mu)|^2}{|\mu|}\, d\mu$$ ← admissibility criterion.

- $s \propto \frac{1}{2^j}$ - compression $0 < s < 1$
  - dialation $s > 1$
- Show Fig 7.16 G+W

Fast Wavelet Transform

$$\phi(x) = \sum_n h_\phi(n)\,\sqrt{2}\,\phi(2x-n)$$

$$x \leftarrow 2^j$$

$$\phi(2^j x - k) = \sum_n h_\phi(n)\,\sqrt{2}\,\phi\left(2(2^j x - k) - n\right)$$

$$= \sum_m h_\phi(m-2k)\,\sqrt{2}\,\phi(2^{j+1} x - m)$$

Similarly

$$\left.\begin{aligned} &\psi(2^j x - k) = \sum_m h_\psi(m-2k)\,\sqrt{2}\,\phi(2^{j+1} x - m) \\ &\text{Recall}\quad d_j(k) = \int f(x)\, 2^{j/2}\, \psi(2^j x - k)\, dx \end{aligned}\right\} \Rightarrow$$

$$d_j(k) = \sum_m h_\psi(m-2k)\, c_{j+1}(m)$$

Similarly

$$c_j(k) = \sum_m h_\phi(m-2k)\, c_{j+1}(m)$$

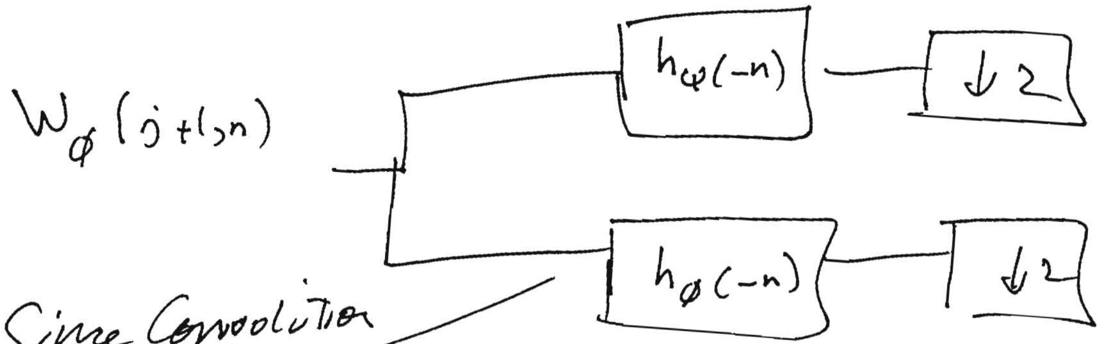
$c_j(k) \longrightarrow W_\phi(j,k)$
$d_j(k) \longrightarrow W_\psi(j,k)$ } DWT

as $f(x) \longrightarrow f(n)$

Then

$$\left.\begin{aligned} W_\psi(j,k) &= \sum_m h_\psi(m-2k)\, W_\phi(j+1,m) \\ W_\phi(j,k) &= \sum_m h_\phi(m-2k)\, W_\phi(j+1,m) \end{aligned}\right\}$$

$$\Longrightarrow \begin{cases} W_\psi(j,k) = h_\psi(-n) * W_\phi(j+1,n)\Big|_{n=2k} & k \geq 0 \\ W_\phi(j,k) = h_\phi(-n) * W_\phi(j+1,n)\Big|_{n=2k} & k \geq 0 \end{cases}$$



— Subband Analysis

Since Convolution
Can use FFT or
other Fast Algorithm

Show Fig 7.18