Time-Frequency – Wavelets

- Stionary v.s. non-stationary signals
- 2D Time frequency representation $S(t,f)$ of signal $x(t)$
  - spectral characteristics depending on time.
- Gabor: defined Short Time Fourie Transform as

$$STFT(\tau, f) = \int x(t)\, g^*(t-\tau)\, e^{-j2\pi f t}\, dt$$

- Fig 1 from Vetterli 1991

1

What is Wrong with S.T.F.T.?

- Given window function $g(t) \longleftrightarrow G(f)$

- Define rms "bandwidth" of g as:

$$\Delta f^2 = \frac{\int f^2 |G(f)|^2 \, df}{\underbrace{\int |G(f)|^2 \, df}_{\to \text{energy of } g.}}$$

$\Delta f \hat{=}$ resolution in frequency of STFT.

- Similarly, spread in time Δt :

$$\Delta t^2 = \frac{\int t^2 |g(t)|^2 \, dt}{\int |g(t)|^2 \, dt}$$

- Two pulses in time can be discreminated if more than Δt apart

- Heizenberg Uncertainty principle.
  - resolution in Time and freq. cannot be arbitrarily small

  time-bandwidth product $\simeq \Delta t \Delta f \geq \frac{1}{4\pi}$

- Gaussian windows meet the bound with equality.

- <u>Observation</u>: once $g(t)$ is chosen for S.T.F.T, Time & freq. resolution given by Δt and Δf is **<u>FIXED</u>** over The entire T.F. plane.

  - Same window is used @ all frequencies.

- Show Fig 2 of Vetterli 1991
- Fig 4.3 of S. Mallat Book.

Heisenberg Box for S.T.F.T.

Assume: g : real, symmetric $g(t) = g(-t)$

$$g_{u,\xi}(t) = e^{j\xi t}\, g(t-u) \qquad \|g\| = 1$$

Energy density or Spectrogram

$$P_S f(u,\xi) = \left| \int_{-\infty}^{+\infty} f(t)\, g(t-u)\, e^{-j\xi t}\, dt \right|^2$$

- Spectrogram measures the energy of f in a T.F. neighborhood (u, ξ) specified by Heisberg box of $g_{u,\xi}$
- Time spread around u is indp. of u, ξ

$$\sigma_t^2 = \int_{-\infty}^{+\infty} (t-u)^2\, |g_{u,\xi}(t)|^2\, dt = \int_{-\infty}^{+\infty} t^2\, |g(t)|^2\, dt$$

$$\hat{g}_{u,\xi}(\omega) = \hat{g}(\omega - \xi)\, e^{-j u(\omega - \xi)}$$

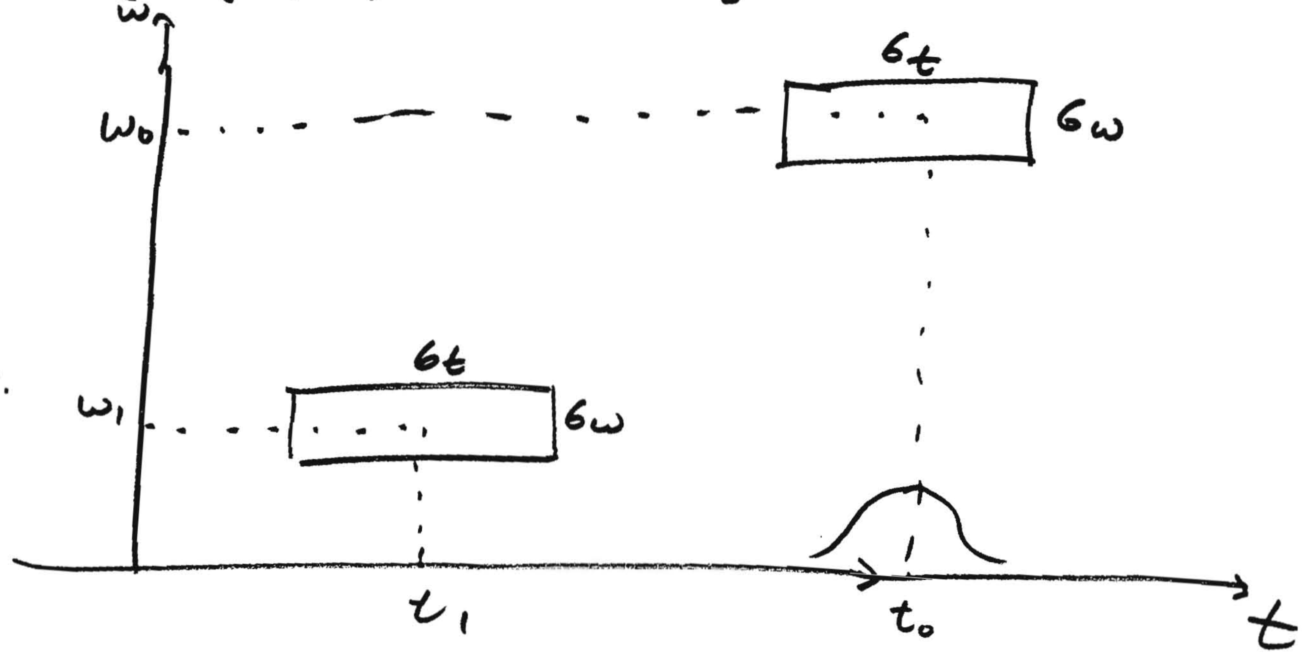
$\Rightarrow$ Freq. spread around ξ is

$$\sigma_\omega^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\omega - \xi)^2 \, |\hat{g}_{u,\xi}(\omega)|^2 \, d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \omega^2 \, |\hat{g}(\omega)|^2 d\omega$$

$\Rightarrow g_{u,\xi}$ corresponds to a Heisberg box of area $\sigma_t \sigma_\omega$ centered @ (u, ξ) $\longrightarrow$ Fig 4.2 of Mallat.

$\rightarrow$

S.T.F.T same resolution everywhere in T.F. plane.



Continous Wavelet Transform

- change The size/support of window To analyze signal structures of diff. sizes. $\longrightarrow$ Wavelet Transform

- Wavelet transform : decompose signal over dialeted & Translated wavelets.

- wavelet $\psi \in L^2(\mathbb{R})$ $\qquad \hat{\psi}$ is Bandpass filter

$$\int_{-\infty}^{+\infty} \psi(t)\, dt = 0 \qquad \|\psi\| = 1$$

- CWT of $f \in L^2(\mathbb{R})$ at Time u, scale s:

$$W f(u,s) = \langle f, \psi_{u,s} \rangle =$$

$$= \left\langle f, \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) \right\rangle$$

$u \in \mathbb{R}$

$s \in \mathbb{R}^+$

$$Wf(u,s) = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{s}} \psi^*\left(\frac{t-u}{s}\right) du$$

$$= f * \check{\psi}_s(u)$$

where $\check{\psi}_s(t) \triangleq \frac{1}{\sqrt{s}} \psi^*\left(\frac{-t}{s}\right)$

show Fig 4.6 & 4.7 of Mallat book 3rd E

Thm: Let $\psi \in L^2(\mathbb{R})$ be a real fn. s.T.

$$C_\psi = \int_0^\infty \frac{|\hat{\psi}(\omega)|}{\omega} d\omega < \infty$$

Any $f \in L^2(\mathbb{R})$ satisfies

$$f(t) = \frac{1}{C_\psi} \int_0^\infty \int_0^\infty Wf(u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) du \frac{ds}{s^2}$$

and $$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^{+\infty} |Wf(u,s)|^2 du \frac{ds}{s^2}$$

Comments:

- $C_\psi < \infty$ is called admissibility condition
- If $\hat{\psi}(0) = 0$ (ie. $\hat{\psi}$ is Bandpass) & $\hat{\psi}(\omega)$ is continously differentiable, Then admissibility is satisfied.
- Can show $\hat{\psi}(\omega)$ is continously differentiable if
$$\int_{-\infty}^{+\infty} (1 + |t|)\, \psi(t)\, dt < \infty$$
i.e sufficient time decay.

~~Suppose~~

Frequency-Time Resolution

Suppose ψ is centered around 0

$\psi_{u,s} = \psi\left(\frac{t-u}{s}\right)$ centered around $t = u$

Can show: $$\int_{-\infty}^{+\infty} (t-u)^2\, |\psi_{u,s}(t)|^2\, dt = s^2 \sigma_t^2$$

where $$\sigma_t^2 = \int_{-\infty}^{+\infty} t^2\, |\psi(t)|^2 dt$$

Assume $\psi \in L^2(\mathbb{R})$ is analytic i.e.

$\hat{\psi}(\omega) = 0$ if $\omega < 0$

$\Rightarrow$ center freq. η of $\hat{\psi}(\omega)$ is

$$\eta = \frac{1}{2\pi} \int_0^{+\infty} \omega |\hat{\psi}(\omega)|^2 \, d\omega$$

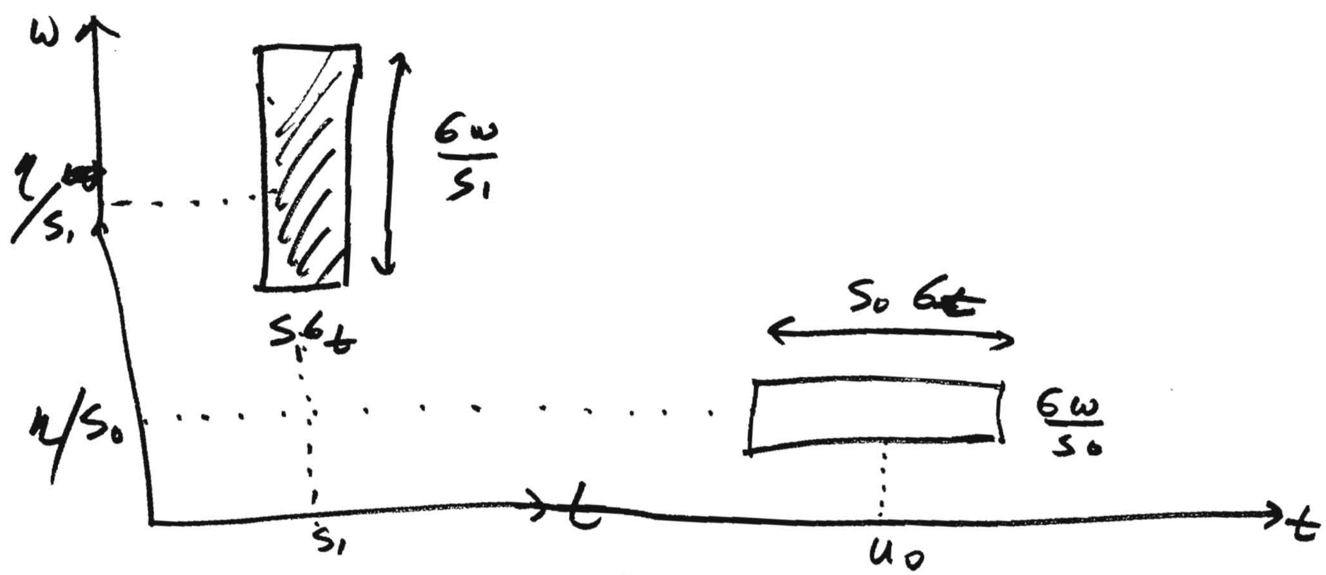
$$\hat{\psi}_{u,s}(\omega) = \sqrt{s}\, \hat{\psi}(s\omega)\, e^{-j\omega u} \longrightarrow \text{center freq } \frac{\eta}{s}$$

Energy spread of $\hat{\psi}(u,s)$ around $\frac{\eta}{s}$ is

$$\frac{1}{2\pi} \int_0^{\infty} \left(\omega - \frac{\eta}{s}\right)^2 |\hat{\psi}_{u,s}(\omega)|^2 d\omega = \frac{\sigma_\omega^2}{s^2}$$

where $$\sigma_\omega^2 = \frac{1}{2\pi} \int_0^{+\infty} (\omega - \eta)^2 \, |\hat{\psi}(\omega)|^2 d\omega$$

<u>Conclusion</u> Energy spread of wavelet T.F. $\psi_{u,s}$ corresponds To heinsberg box centered at $\left(u, \frac{\eta}{s}\right)$ of size $s\sigma_t$ along time $\frac{\sigma_\omega}{s}$ along frequency



- Show Fig 4.9 of Mallat.
- Area of Rectangle = $\sigma_t \sigma_\omega$ → everywhere at all scales.
- local T.F. energy density $P_W f$
  - measures energy of f in Heisberg box of each wavelet $\psi_{u,s}$ centered at $(u, \frac{\eta}{s})$

$$P_W f(u,\xi) = |W f(u,s)|^2 = \left|W f\left(u, \frac{\eta}{\xi}\right)\right|^2$$

This energy density is called scalogram.

Show Fig 4.11 of Mallat

Frames and Reisz Bases

- Consider ~~vector~~ Hilbert space H, vector f in H.

Def ~~R~~ Frame and Reisz Basis:

Sequence $\{\phi_n\}_{n \in \Gamma}$ is a frame of H if

$\exists$ 2 constants $B \geq A > 0$ s.t.

$$\forall f \in H \qquad A \|f\|^2 \leq \sum_{n \in \Gamma} |\langle f, \phi_n \rangle|^2 \leq B \|f\|^2$$

when $A = B \longrightarrow$ frame is Tight.

If $\{\phi_n\}_{n \in \Gamma}$ linearly independent $\longrightarrow$ frame is not redundant $\Longrightarrow$ Riesz basis

Multi-Resolution Approximation

- Approx of a fn at resolution 2^{-j} $\triangleq$ orthogonal projection of fn on space $V_j \subset L^2(\mathbb{R})$

- $V_j \triangleq$ space of all possible approx @ resolution 2^{-j}

- Orthogonal projection of f is $f_j \in V_j$ s.th. $\|f - f_j\|$ is minimized

<u>Def</u> <u>MultiResolutions</u>: Sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ is MRA if these properties hold

(1) ~~Shift~~ Shift invariance

$$\forall (j,k) \in \mathbb{Z}^2 \qquad f(t) \in V_j \iff f(t - 2^j k) \in V_j$$

(2) Sequence of Embedded Subspaces

$$\forall j \in \mathbb{Z} \qquad V_{j+1} \subset V_j$$

(3) Scale Invariance

$$\forall j \in \mathbb{Z} \quad f(t) \in V_j \iff f\left(\frac{t}{2}\right) \in V_{j+1}$$

(4) Downward Completeness:

$$\lim_{j \to \infty} V_j = \bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$$

(5) Upward Completeness

$$\lim_{j \to -\infty} V_j = \text{Closure}\left(\bigcup_{j=-\infty}^{+\infty} V_j\right) = L^2(\mathbb{R})$$

(6) $\exists \; \theta$ s.th. $\{\theta(t-n)\}_{n \in \mathbb{Z}}$ is a Reisz basis of V_0