Pyramid Coding

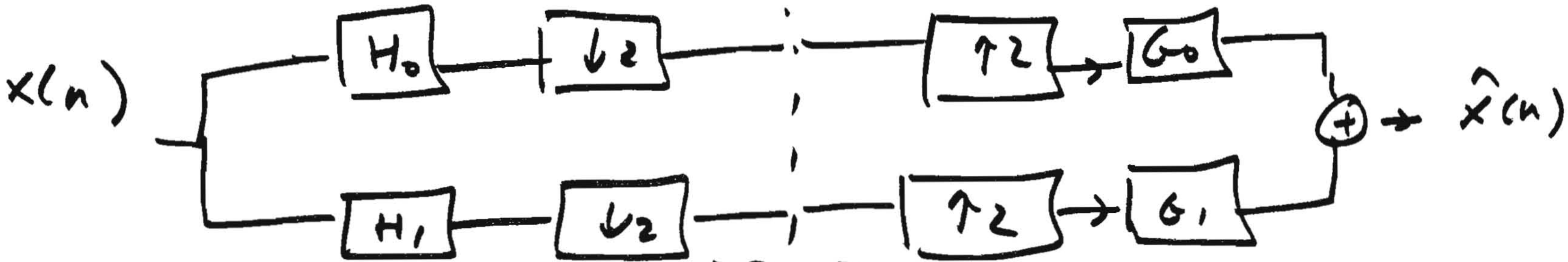
Fig 7.2 G+W 3E :

Total # of pixels in a P+1 level pyramid for $P > 0$

- $N^2\left(1 + \frac{1}{(4)^1} + \frac{1}{(4)^2} + \frac{1}{(4)^3} + \cdots + \frac{1}{4^P}\right) \leq \frac{4}{3}N^2$

- Assumes base level is N^2 or $N \times N$

Subband Coding



$$\hat{X}(\omega) = \frac{1}{2}\overbrace{\left[H_0(\omega)G_0(\omega) + H_1(\omega)G_1(\omega)\right]}^{\text{PR Term}} X(\omega) + \frac{1}{2}\underbrace{\left[H_0(\omega-\pi)G_0(\omega) + H_1(\omega-\pi)G_1(\omega)\right]}_{\text{aliasing Term}} X(\omega-\pi)$$

$$\hat{X}(z) = \frac{1}{2}\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right] X(z) + \frac{1}{2}\left[H_0(-z)G_0(z) + H_1(-z)G_1(z)\right] X(-z)$$

To Eliminate Aliasing, set

$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$$

or

$$H_0(\omega-\pi)G_0(\omega) + H_1(\omega-\pi)G_1(\omega) = 0 \qquad (*)$$

- Can satisfy (*) by selecting

$G_0(\omega) = H_1(\omega - \pi)$ $\qquad$ $G_1(\omega) = -H_0(\omega - \pi)$

Conditions for removing aliasing Term.

- Assume Further That

$H_0(\omega) = H(\omega)$

$H_1(\omega) = H(\omega - \pi)$

Then prototype filter $H(\omega)$ is enough To design all 4 filters:

$$\left.\begin{aligned} H_0(\omega) &= H(\omega) \\ H_1(\omega) &= H(\omega - \pi) \\ G_0(\omega) &= H(\omega) \\ G_1(\omega) &= -H(\omega - \pi) \end{aligned}\right\} \text{Eqn (1)}$$

How about Perfect reconstruction property?

- Need to make the PR term be z^{-k}:

$$\frac{1}{2}\left[H_0(z)\, G_0(z) + H_1(z)\, G_1(z) \right] = z^{-k}$$

$\Rightarrow$ means output is delayed version of input.

- Assume "one" prototype filter $H(\omega)$ in Eqn 1

then:

$$H^2(\omega) - H^2(\omega - \pi) = 2e^{-j\omega k}$$

~~$\Rightarrow$ For PR $H(\omega)$ must satisfy~~

$$\cancel{\left| H^2(\omega) - H^2(\omega-\pi) \right| =}$$

or

$$H^2(z) - H^2(-z) = 2z^{-k} \quad \text{for some } k.$$

Assume $H(\omega)$ is linear phase i.e.

$$H(\omega) = H_r(\omega)\, e^{-j\omega\frac{(N-1)}{2}} \qquad N = \text{filter length.}$$

$$\frac{\hat{X}(\omega)}{X(\omega)} = \left[\, |H(\omega)|^2 - (-1)^{N-1}\, |H(\omega-\pi)|^2 \,\right] e^{-j\omega(N-1)}$$

$\Rightarrow$ delay of overall Filter : $N-1$

Magnitude of overall filter :

- $M(\omega) = |H(\omega)|^2 - (-1)^{N-1} |H(\omega-\pi)|^2$
  - N odd $\Rightarrow$ $M(\frac{\pi}{2}) = 0 \longrightarrow$ Bad!
  - N even $\Rightarrow$ $M(\omega) = |H(\omega)|^2 + |H(\omega-\pi)|^2$

Ideally we want: $M(\omega) = \boxed{1 = |H(\omega)|^2 + |H(\omega-\pi)|^2}$

only soln: $|H(\omega)|^2 = \cos^2 a\omega \longrightarrow$ Trivial

$\Rightarrow$ any nontrivial ~~soln~~ linear phase soln $\longrightarrow$ amplitude distortion

$\Rightarrow$ Impossible To have:

FIR, PR, linear phase, power-complimentary, filters

- Settle with Near Perfect Reconstruction.
- solve an optimization problem
  - Make $M(\omega)$ as flat as possible while minimizing stop band energy of $H(\omega)$

$$J = \alpha \int_{\omega_s}^{\pi} |H(\omega)|^2 d\omega + (1-\alpha) \int_0^{\pi} \left(M(\omega) - 1\right)^2 d(\omega)$$

- optimize J. w.r.t. To Filter taps. subject To impulse response being symmetric $\Rightarrow$ i.e. linear phase.

<u>Def</u>: Power complimentary $|H_0(\omega)|^2 + |H_1(\omega)|^2 = 1$

Consider The Case of 2 Prototype filters

$G_0(z) = H_1(-z)$ $\qquad G_1(z) = -H_0(-z)$

Define $P_0(z) = H_0(z)\, G_0(z)$

Then PR Condition becomes:

$$P_0(z) - P_0(-z) = 2 z^{-l} \qquad l = \text{int.}$$

or

$$z^{l} P_0(z) - z^{l} P_0(-z) = 2$$

Define $P(z) = z^{l} P_0(z)$

- Can show $P(z)$ is symmetric polynomial.
- Write $P(z)$ as:

$$P(z) = 1 + p_1 (z + z^{-1}) + p_3 (z^{3} + z^{-3}) + p_5 (z^{5} + z^{-5}) + \ldots$$

- In compression, want max # of zeros of $H_0(z)$ to be at $z = -1$ or $\omega = \pi$
- $z = -1$ also zero of $P_0(z)$ and $P(z)$

Write $P(z)$ as

$$P(z) = (1+z^{-1})^m (1+z)^m R(z)$$

where $R(z)$ is symmetric polynomial. $R(z) = R(z^{-1})$

- Suppose $R(z)$ is of the form:

$$\boxed{R(z) = r_0 + \sum_{s=1}^{m-1} r_s (z^s + z^{-s})} \qquad \text{Eqn 2}$$

- Suppose $R(z) = \frac{1}{2}$ $m=1$ $\Rightarrow$

$$P(z) = \frac{1}{2}(z + 2 + z^{-1}) = \frac{1}{2} z(1+z^{-1})(1+z^{-1})$$

$$= z^{l} H_0(z) G_0(z)$$

choose $H_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1})$

- lowest order $G_0(z)$ is for $l=1$ $\Rightarrow$

$$G_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1}) \Rightarrow \text{Haar Filters}$$

Suppose $m=2$ in Eqn 2 and

$$R(z) = az + b + az^{-1}$$

Then

$$P(z) = (1+z^{-1})^2 (1+z)^2 (az + b + az^{-1})$$

$$= az^3 + (4a+b)z^2 + (7a+4b)z + (8a+6b)$$

$$+ (7a+4b)z^{-1} + (4a+b)z^{-2} + az^{-3}$$

- Even powers of $P(z)$ are zero (symmetric)
- coeff of $z^0 = 1$

$\Rightarrow$

$4a+b=0 \quad 8a+6b=1 \Rightarrow a = -\frac{1}{16} \quad b = \frac{1}{4}$

$$\Rightarrow \boxed{P(z) = \frac{1}{16} z^3 (1+2z^{-1}+z^{-2})^2 (-1+4z^{-1}-z^{-2})} \quad \text{Eqn 3}$$

- Recall $P(z) = z^{\ell} P_0(z) = z^{\ell} G_0(z) H_0(z)$.

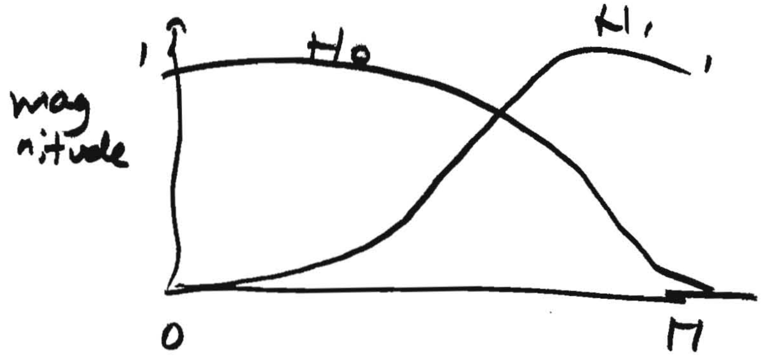
How about factorizing $P(z)$ with $\ell = 3$?

(a): $H_0(z) = \frac{1}{2}(1 + \frac{1}{2} 2z^{-1} + z^{-2})$

$$G_0(z) = \frac{1}{8}(1+2z^{-1}+z^{-2}) \times (-1+4z^{-1}-z^{-2}) = \frac{1}{8}(-1+2z^{-1}+6z^{-2}+2z^{-3}-z^{-4})$$

Corresponding hi pass filters are

$H_1(z) = G_0(-z)$ $\quad$ $G_1(z) = -H_0(-z)$



- Le Gall 3/5 Tap filter pair

(b) Another factorization of Eqn 3 is:

$$H_0(z) = \frac{1}{8}\left(-1 + 2z^{-1} + 6z^{-2} + 2z^{-3} - z^{-4} \right]$$

$$G_0(z) = \frac{1}{2}\left(1 + 2z^{-1} + z^{-2}\right)$$

$$H_1(z) = \frac{1}{2}\left(1 - 2z^{-1} + z^{-2}\right)$$

$$G_1(z) = \frac{1}{8}\left(1 + 2z^{-1} - 6z^{-2} + 2z^{-3} + z^{-4}\right)$$

Le Gall 5/3 Tap filter.

(c) Another factorization of Eqn 3: $l=3$

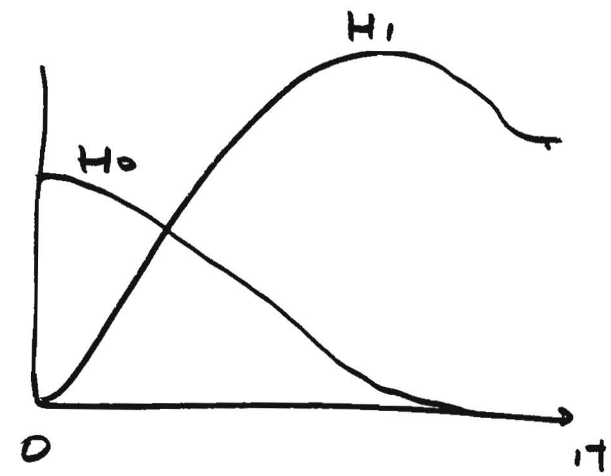
$$H_0(z) = \frac{1}{8}\left(1 + 3z^{-1} + 3z^{-2} + z^{-3}\right)$$

$$G_0(z) = \frac{1}{2}\left(-1 + 3z^{-1} + 3z^{-2} - z^{-3}\right)$$

$$H_1(z) = \frac{1}{2}\left(-1 - 3z^{-1} + 3z^{-2} + z^{-3}\right)$$

$$G_1(z) = \frac{1}{8}\left(-1 + 3z^{-1} - 3z^{-2} + z^{-3}\right)$$

Called Daubachie 4/4 tap filter pair.



Orthogonal Filter banks

- What if we require output of filter bank to be orthogonal Transformation of input signal?
- So far we have PR, Linear Phase, FIR but non-orthogonal $\longrightarrow$ bi-orthogonal.
- Now we want orthogonal FIR, PR
  $\longrightarrow$ becomes nonlinear phase!
- Recall $P(z)$ is symmetric polynomial.
  $\Rightarrow$ factors of the form $(\alpha z+1)(1+\alpha z^{-1})$
  $\longrightarrow$ One strategy: assign $(1+\alpha z^{-1})$ to $H_0(z)$
  $z^{-1}(\alpha z+1)$ to $G_0(z)$

$\Longrightarrow$ $G_0(z) = z^{-N} H_0(z^{-1})$

If coeff real $\Longrightarrow$ $G_0(\omega) = H_0(-\omega)$
$\Longrightarrow$ same magnitude response

- Desirable to have filters with as many zeros of $P(z)$ as possible. at $z=-1$

Choose $P(z) = (1+z^{-1})^m (1+z)^m R(z)$

- Possibility:
  - Assign all factors of $P(z)$ having zeros outside unit circle to $G_0(z)$.
    inside " " to $H(z)$
  - can show zeros of $P(z)$ on unit circle have even multiplicity $\Rightarrow$ ½ go to H_0
    ½ " " H_1

$\Longrightarrow$ $H_0(z)$ minimum phase filter
$G_0(z)$ max phase filter

Ex: Factorize $P(z)$ from Eqn 3 this way:
The factor $(1-4z^{-1}+z^{-2})$ has a zero inside the unit circle @ $z=2-\sqrt{3}$ and another zero outside the unit circle at $z=2+\sqrt{3}$ $\Longrightarrow$

$\Rightarrow$ minimum phase spectral factor of $P(z)$.

$$H_0(z) = \frac{1}{4(\sqrt{3}-1)} (1+z^{-1})^2 \left(1-(2-\sqrt{3})z^{-1}\right)$$

$$= 0.48 + 0.83\, z^{-1} + 0.22\, z^{-2} - 0.13\, z^{-3}$$

Max phase spectral factor:

$$G_0(z) = z^{-3} H_0(z^{-1}) = -0.129 + 0.22\, z^{-1} + 0.83\, z^{-2} + 0.48\, z^{-3}$$

More generally:

Suppose $H_0(z)$ is FIR filter of order N satisfying "power symmetry" condition:

$$H_0(z)\, H_0(z^{-1}) + H_0(-z)\, H_0(-z^{-1}) = 1$$

choose $H_1(z) = z^{-N} H_0(-z^{-1})$ $\Rightarrow$

Then $H_0(z) H_1(-z) - H_0(-z) H_1(z) =$ PR condition $=$

$$= -z^{-N}\left[H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1})\right] = -z^{-N}$$

Wow!! PR condition has been satisfied with delay of $-N$ and gain of -1!

$$\Longrightarrow \left.\begin{aligned} H_1(z) &= z^{-N} H_0(-z^{-1}) \\ G_0(z) &= z^{-N} H_0(z^{-1}) \\ G_1(z) &= z^{-N} H_1(z^{-1}) \end{aligned}\right\}$$

$\Longrightarrow$ Perfect Recon, power symmetric Filter bank $\longrightarrow$ Also called orthogonal Filter bank.

If $H_0(z)$ causal FIR $\Longrightarrow$ 3 other filters causal

- $|G_i(\omega)| = |H_i(\omega)|$ $i = 1, 2$
- $|H_1(\omega)| = |H_0(-\omega)|$; if real coeff in transfer function and $H_0(z)$ ~~the~~ is LPF $\Longrightarrow$ H_1 is HPF.

⇒ Conclusion: To design orthogonal Filter bank, all we have to do is to design power symmetric LPF $H_0(z)$.

- Two steps:
  1. Design $P_0(z) = H_0(z)\,H_0(z^{-1})$
  2. spectrally factor $P_0(z)$ to get $H_0(z)$.

Show Figs 7.8 + 7.9 of G+W