EE243 Advanced Electromagnetic Theory

Lec # 26 Review for Final Exam

- **Final Exam Specification Sheet (see handout)**
  - **(Gives sections of Jackson, Kogenik and Harrington)**
- **Guided Waves**
  - **Lec 13-14; HW 7.1-7.2;**
- **Dielectric, Corrugated and Plasmon Waveguides**
  - **Lec 15-20; HW 7.3, 8.1-8.3**
- **Radiation and Scattering**
  - **Lec 21-25; HW 9.1-9.3**

Reading: Summarized on Final Exam Specification Sheet

Magnetic/Electric Duality

Harrington 3.2

Electric Sources

$$\nabla \times \overline{H} = -i\omega\varepsilon\overline{E} + \overline{J}$$

$$\nabla \times \overline{E} = i\omega\mu\overline{H}$$

$$\overline{H} = \nabla \times \overline{A}$$

$$\overline{\mathrm{A}} = \frac{1}{4\pi}\int_V \frac{\overline{J}e^{ik|\overline{x}-\overline{x}'|}}{|\overline{x}-\overline{x}'|} d^3x'$$

Magnetic Sources

$$\nabla \times \overline{E} = i\omega\mu\overline{H} - \overline{M}$$

$$\nabla \times \overline{H} = -i\omega\varepsilon\overline{E}$$

$$\overline{E} = -\nabla \times \overline{F}$$

$$\overline{\mathrm{F}} = \frac{1}{4\pi}\int_V \frac{\overline{M}e^{ik|\overline{x}-\overline{x}'|}}{|\overline{x}-\overline{x}'|} d^3x'$$

- Dual equations for problems in which only an electric source J or only a magnetic source M are present.

Source Free Region

Harrington Strategy 3.12

$$\nabla^2 \overline{A} + k^2 \overline{A} = 0$$

$$\nabla^2 \overline{F} + k^2 \overline{F} = 0$$

$$\overline{E} = -\nabla \times \overline{F} + i\omega\mu\overline{A} + \frac{1}{i\omega\varepsilon}\nabla\left(\nabla \cdot \overline{A}\right)$$

$$\overline{H} = \nabla \times \overline{A} + i\omega\overline{F} + \frac{1}{i\omega\mu}\nabla\left(\nabla \cdot \overline{F}\right)$$

- With the Lorenz gauge A and F satisfy the wave equation and the fields are give by above equations.
- Choosing the vectors A and B to only be in the z direction is adequate.
- Each potentially contributes 5 components of the E,H combination.

Vector Potential in z Direction

$$\bar{A} = \psi\hat{z}$$

$$E_x = \frac{1}{-i\omega\varepsilon}\frac{\partial^2\psi}{\partial x\partial z}$$

$$E_y = \frac{1}{-i\omega\varepsilon}\frac{\partial^2\psi}{\partial y\partial z}$$

$$E_z = \frac{1}{-i\omega\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi$$

$$H_x = \frac{\partial\psi}{\partial y}$$

$$H_y = -\frac{\partial\psi}{\partial x}$$

$$H_z = 0$$

$$\bar{F} = \psi\hat{z}$$

$$E_x = -\frac{\partial\psi}{\partial y}$$

$$E_y = \frac{\partial\psi}{\partial x}$$

$$E_z = 0$$

$$H_x = \frac{1}{-i\omega\mu}\frac{\partial^2\psi}{\partial x\partial z}$$

$$H_y = \frac{1}{-i\omega\mu}\frac{\partial^2\psi}{\partial y\partial z}$$

$$H_z = \frac{1}{-i\omega\mu}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi$$

Jackson Strategy Eq. 8.26

$$\left[\nabla_t^2 + \left(\mu\varepsilon\omega^2 - k^2\right)\right]\overline{E} = 0$$ Jackson 8.2

$$\overline{E} = E_z\hat{z} + \overline{E}_t$$

$$\overline{E}_t = \frac{1}{\left(\mu\varepsilon\omega^2 - k^2\right)}\left[k\nabla_t E_z - \omega\hat{z}\times\nabla_t B_z\right]$$

$$\overline{B}_t = \frac{1}{\left(\mu\varepsilon\omega^2 - k^2\right)}\left[k\nabla_t B_z + \mu\varepsilon\omega\hat{z}\times\nabla_t E_z\right]$$

- E and B satisfy wave equation with transverse operator and $-k^2$
- Break up E and B into longitudinal and transverse
- Transverse fields can be found from E_z and B_z.

Waveguide Simplifications (Revised)

$$\overline{E}_t = \pm\frac{ik}{\gamma^2}\nabla_t \overline{E}_z$$

$$\overline{B}_t = \pm\frac{ik}{\gamma^2}\nabla_t \overline{B}_z$$

$$\gamma^2 = \mu\varepsilon\omega^2 - k^2$$

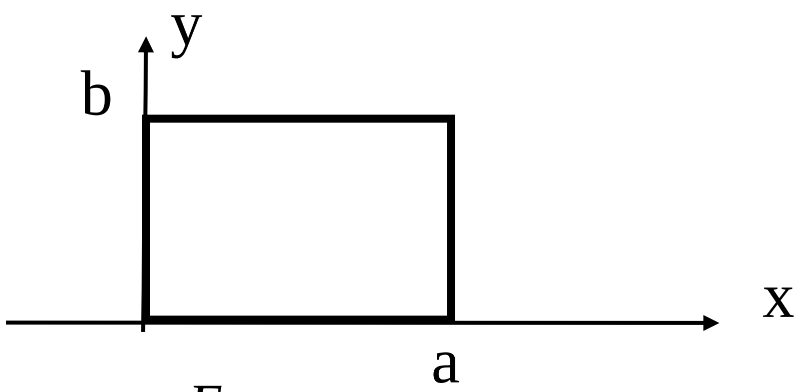
$$\overline{H}_t = \frac{\pm 1}{Z}\hat{z}\times\overline{E}_t$$

$$Z_{TM} = \frac{k}{\omega\varepsilon} = \frac{k}{k_0}\sqrt{\frac{\mu}{\varepsilon}}$$

$$Z_{TE} = \frac{\mu\omega}{k} = \frac{k_0}{k}\sqrt{\frac{\mu}{\varepsilon}}$$

- Set Boundary Condition
  - If TE Ez = 0 on p.e.c. sidewall.
  - If TM use 8.26 to get normal derivative of Bz = 0
- Solve for Ez and/or Bz
- Then use gradient to find the associated transverse B or E
- Use impedance to find the associated transverse E and B (or use 8.26)

Rectangular Waveguide Example (TM)



$$\psi = E_z$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2 \right) \psi = 0$$

$$\psi \,|_S = 0$$

$$\overline{E}_t = \pm \frac{ik}{\gamma^2} \nabla_t \psi$$

$$\overline{H}_t = \frac{\pm 1}{Z} \hat{z} \times \overline{E}_t$$

$$Z_{TM} = \frac{k}{\omega \varepsilon} = \frac{k}{k_0} \sqrt{\frac{\mu}{\varepsilon}}$$

$$E_{zmn} = E_0 \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b})$$

$$\gamma_{mn}{}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$$

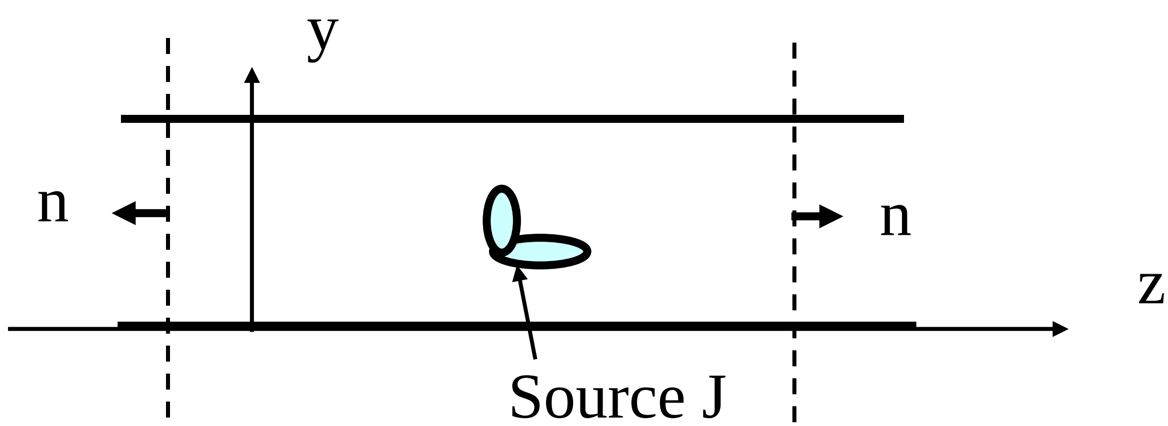
$$E_x = E_0 \frac{ik\pi}{\gamma_{mn}{}^2 a} \cos(\quad) \sin(\quad)$$

$$E_y = E_0 \frac{ik\pi}{\gamma_{mn}{}^2 b} \sin(\quad) \cos(\quad)$$

$$H_x = -E_0 \frac{ik\pi}{Z_{TM} \gamma_{mn}{}^2 b} \sin(\quad) \cos(\quad)$$

$$H_y = E_0 \frac{ik\pi}{Z_{TM} \gamma_{mn}{}^2 a} \cos(\quad) \sin(\quad)$$

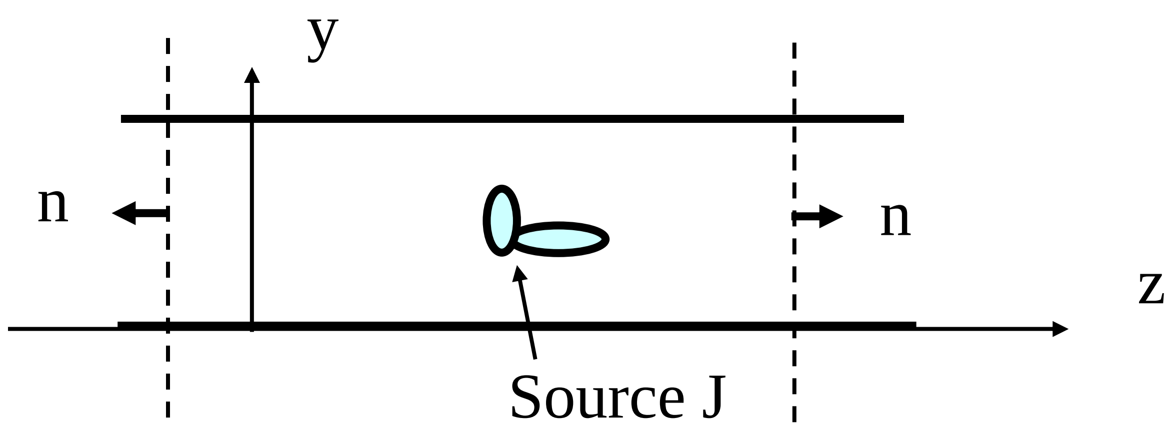
Fields Generated by a Localized Source



To find amplitude of a given mode propagating to the right

- General expansion for modes to right
- Assume the given mode of arbitrary amplitude is propagating to left from outside the boundary
- Apply reciprocity to the volume for these two sets of fields

Fields Generated by a Localized Source (Cont.)



Apply reciprocity to the volume for these two sets of fields

- Integral over wall is zero
- Integral over left cut-plane is zero as all modes going same direction
- Integral over right cut plane is proportional to outgoing mode amplitude times incoming mode amplitude
- Intergral over the source measures the component of the source with the x,y eigenfunction variation as well as phase coherence with z and is proportional to incoming mode amplitude
- Ratio cancels incoming amplitude and gives outgoing amplitude.

Representation of Fields in Guide

$$\overline{E}^{+} = \sum_{\lambda} A_{\lambda}^{+}\left[\overline{E}_{t\lambda}(x,y) + \overline{E}_{z\lambda}(x,y)\right]e^{-ik_{\lambda}z}$$

$$\overline{H}^{+} = \sum_{\lambda} A_{\lambda}^{+}\left[\overline{H}_{t\lambda}(x,y) + \overline{H}_{z\lambda}(x,y)\right]e^{-ik_{\lambda}z}$$

$$\overline{E}^{-} = \sum_{\lambda} A_{\lambda}^{-}\left[\overline{E}_{t\lambda}(x,y) - \overline{E}_{z\lambda}(x,y)\right]e^{+ik_{\lambda}z}$$

$$\overline{H}^{-} = \sum_{\lambda} A_{\lambda}^{-}\left[-\overline{H}_{t\lambda}(x,y) + \overline{H}_{z\lambda}(x,y)\right]e^{+ik_{\lambda}z}$$

$$\overline{E}_{TEST}^{-} = C_{\lambda}^{-}\left[\overline{E}_{t\lambda}(x,y) - \overline{E}_{z\lambda}(x,y)\right]e^{+ik_{\lambda}z}$$

$$\overline{H}_{TEST}^{-} = C_{\lambda}^{-}\left[-\overline{H}_{t\lambda}(x,y) + \overline{H}_{z\lambda}(x,y)\right]e^{+ik_{\lambda}z}$$

- Localized source J creates waves
  - Index λ goes over TE, TM, m, n
  - To right of source only waves to +z and sum over all TE and TM modes that propagate
  - To left of source only waves to –z and sum over all TE and TM waves
  - To left fields have signs altered div E = div H = 0
  - Test wave from outside going to left across volume

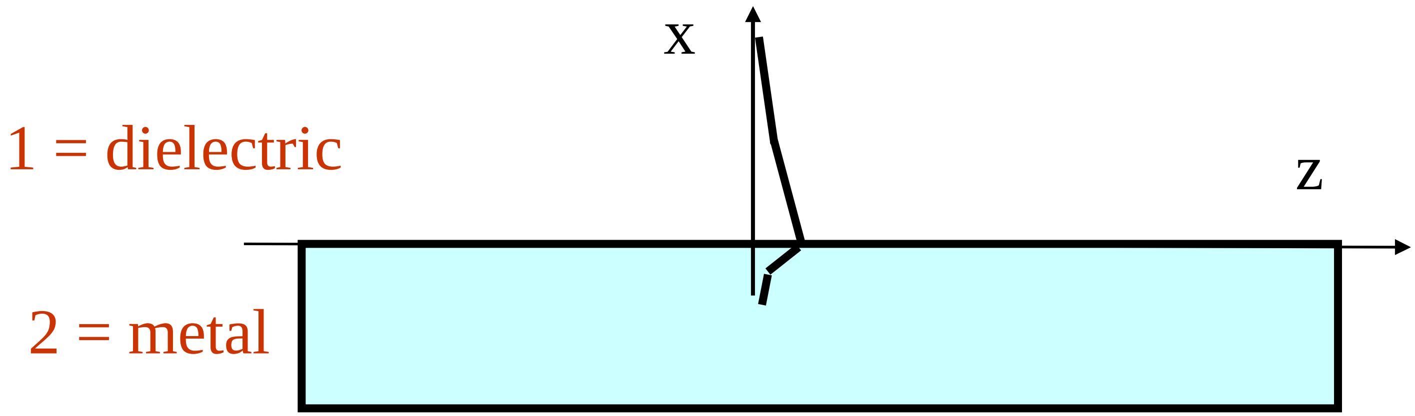
Apply Reciprocity Formulation

$$\nabla \cdot \left(\overline{E}_{TEST} \times \overline{H}_{\lambda}^{\pm} - \overline{E}_{\lambda}^{\pm} \times H_{TEST} \right) = \overline{J} \cdot \overline{E}_{TEST}$$

$$\int_S \left(\overline{E}_{TEST} \times \overline{H}_{\lambda}^{\pm} - \overline{E}_{\lambda}^{\pm} \times H_{TEST} \right) \cdot \hat{n} da = \int_V \overline{J} \cdot \overline{E}_{TEST} d^3 x$$

- Source J produces the modes leaving the localized source region with amplitudes A_λ
- Source free TEST wave enters the volume and takes a measure of E
- Take Poynting Theorem like interaction

Are There Waves on Material Surfaces?



$$H_{1y} = H_1 \hat{y} e^{-v_1 x} e^{i k_z z}$$

$$H_{2y} = H_2 \hat{y} e^{+v_2 x} e^{i k_z z}$$

$$v_1 = \sqrt{k_z^2 - \omega^2 \mu_0 \varepsilon_1}$$

$$v_2 = \sqrt{k_z^2 - \omega^2 \mu_0 \varepsilon_2}$$

- Consider TM w/r z case with Hy given and same z phase variation
- Will have Hy, Ez and Ex (but Ey = Hx = Hz = 0)

Boundary Conditions

$$E_{1z} = \frac{-v_1}{-i\omega\varepsilon_1} H_{1y}$$

$$E_{2z} = \frac{+v_2}{-i\omega\varepsilon_2} H_{2y}$$

$$\frac{-iv_1}{\varepsilon_1} = \frac{+iv_2}{\varepsilon_2}$$

$$\frac{E_{1z}}{H_{1y}} = \frac{E_{2z}}{H_{2y}}$$

$$-Z_{+x} = \frac{-v_1}{-i\omega\varepsilon_1} = \frac{+v_2}{-i\omega\varepsilon_2} = Z_{-x}$$

- Hy continuous (or) D normal continuous gives $H_{10} = H_{20}$.
- Ez continuous gives final constraint to find k_z.
- This constraint is the same as setting the impedance looking upward equal to the negative of the impedance looking downward.
- Impedance looking upward is capacitive (neg imy).
- Impedance looking downward thus need to be inductive.

Solving for Surface Wave Conditions

$$\frac{v_1}{\varepsilon_1} = \frac{-v_2}{\varepsilon_2}$$

$$v_1 = \sqrt{k_z^2 - \omega^2 \mu_0 \varepsilon_1}$$

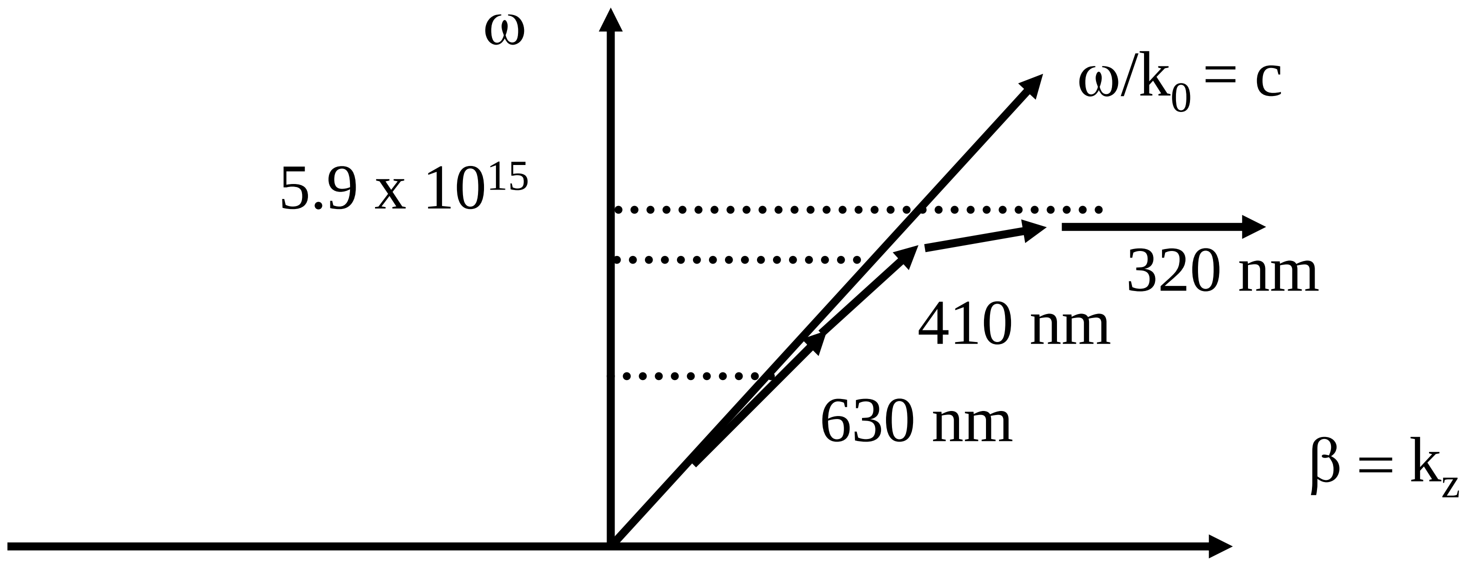
$$v_2 = \sqrt{k_z^2 - \omega^2 \mu_0 \varepsilon_2}$$

$$k_z = k_1 \sqrt{\frac{\varepsilon_2}{(\varepsilon_2 + \varepsilon_1)}}$$

$$v_1 = k_1 \sqrt{\frac{-\varepsilon_1}{(\varepsilon_2 + \varepsilon_1)}}$$

- Constraint
- Substitute definition of v_1 and v_2 to solve for k_z.
- Substitute solution for k_z to find other properties
  - v_1 and v_2 (localization in x)
  - Resolution in z with large k_z
  - Probe height in x

ω–β Diagram for Plasmon



$$f = 2\pi\left(3\cdot10^{17}\,nm/s\right)/\lambda_{nm}$$

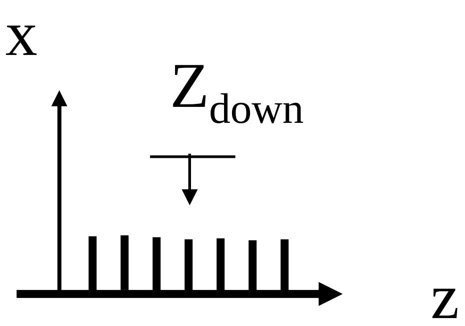
$$f = \left(1.89\cdot10^{18}\,nm/s\right)/320_{nm} = 5.9\cdot10^{15}$$

- The plasmons start as frequency is increased
  - close to the speed of light line,
  - become slightly slower, and
  - turn into a very slow wave (horizontal line) at the plasma frequency.

Surface Topography Can Aid Guided Waves

Harrington 4.8

Corrugated Surface



Impedance looking down into the corrugations is inductive.

$$Z_{down} = -i\sqrt{\frac{\mu_1}{\varepsilon_1}}\tan k_0 d = -i377\tan k_0 d$$

$$Z_{up} = \frac{-v_1}{i\omega\varepsilon_1}$$

$$k_z = k_0\sqrt{1+\tan^2 k_0 d}$$

- Impedance looking into slot is that of a parallel plate waveguide terminated in a short.
- Slsot must be narrow compared to a wavelength
- Depth must be >5% of wavelength to contribute.
- For plasmons
  - effects might add
  - which wavelength should be used

Dielectric Waveguides

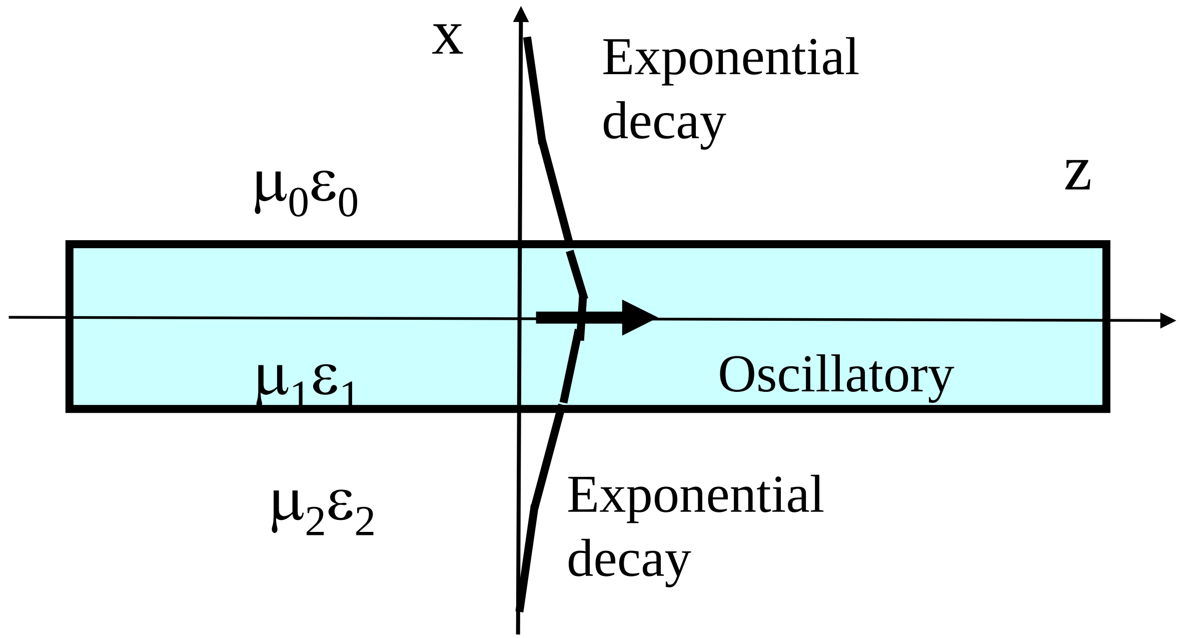
$$e^{j\omega t}$$

$$e^{-jk_z z}$$

$$v_0 = \sqrt{k_z^2 - \omega^2 \mu_0 \varepsilon_0}$$

$$k_x = \sqrt{\omega^2 \mu_0 \varepsilon 1 - k_z^2}$$

$$v_2 = \sqrt{k_z^2 - \omega^2 \mu_2 \varepsilon_2}$$



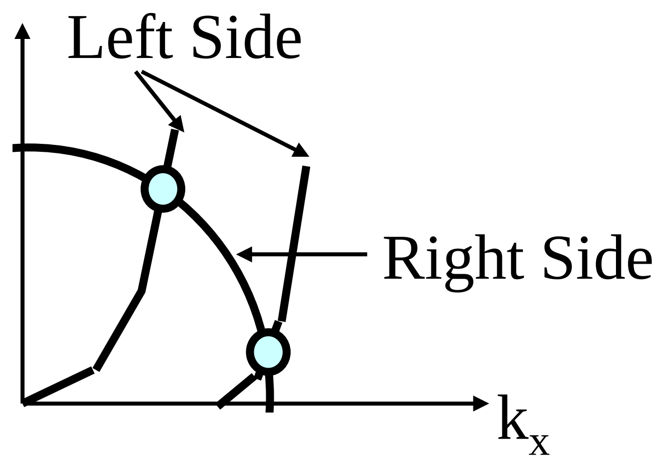
- Three regions
- Choose TM (or TE)
- Will have Hy, Ez and Ex (Ey, Hx, and Hz)

Dielectric Waveguide: Physical Nature

Harrington 4.7 Special case of air on top and bottom, thickness a

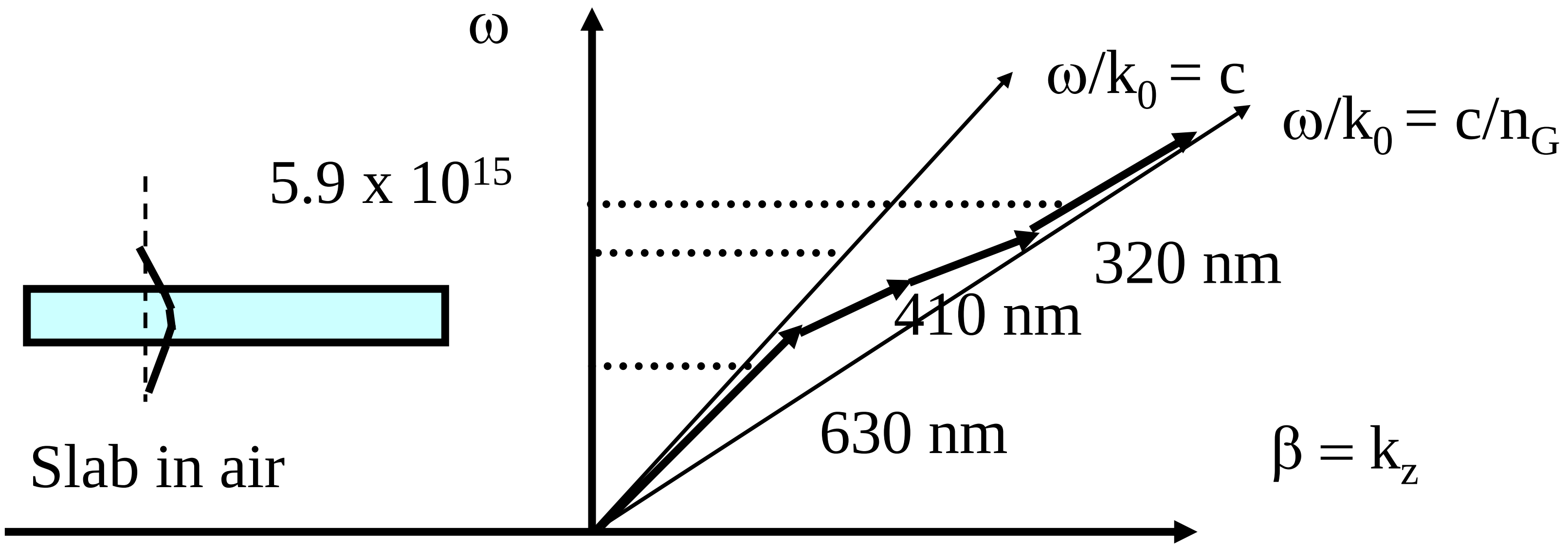
TM

odd $$\frac{k_x a}{2}\tan\frac{k_{x0} a}{2} = \frac{\varepsilon_1}{\varepsilon_0}\frac{v_0 a}{2}$$



- Right hand side is a circle; Left hand side is spikes in tan (See H Fig 4-11)
- Odd sin(k_yx) variations have no cut-off (always exist) in both TM and TE
- Mutiple solutions (intersections) give multiple modes
- Additional new mode about every half wavelength of oscillatory variation.
- Weighted by material contrast sqrt ($\mu_1\varepsilon_1$-$\mu_0\varepsilon_0$)

ω–β Diagram for Dielectric Guide

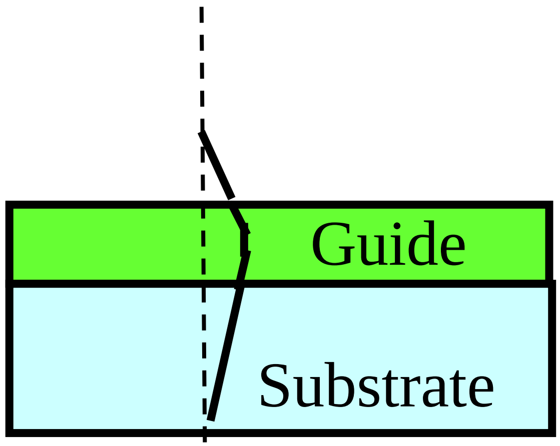


$$f = 2\pi\left(3\cdot 10^{17}\, nm/s\right)/\,\lambda_{nm}$$

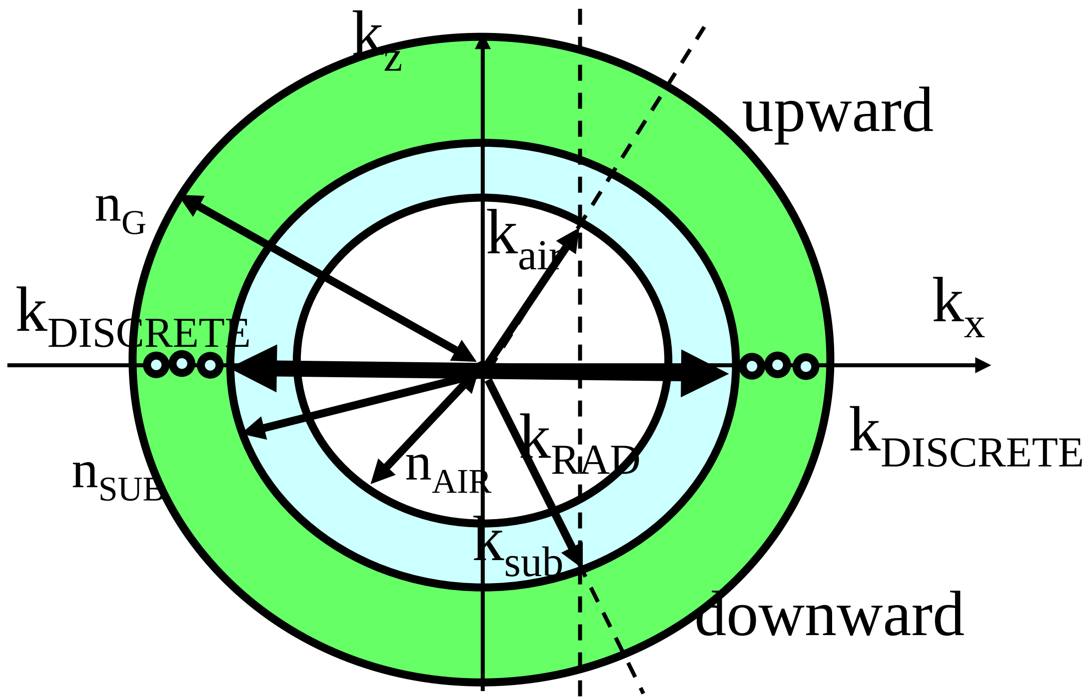
$$f = \left(1.89\cdot 10^{18}\, nm/s\right)/\,320_{nm} = 5.9\cdot 10^{15}$$

- The mode may starts along n_{AIR} at low frequency
- Then transitions toward the n_{GUIDE}
- And asymptotes to N_{GUIDE}

Dielectric Layer Modes



$n_{GUIDE} > N_{SUB} > N_{AIR}$



- Discrete guided modes
- Continuum of radiating modes in air and substrate
- TE and TM cases not separated

Orthogonality of Modes (Cont.)

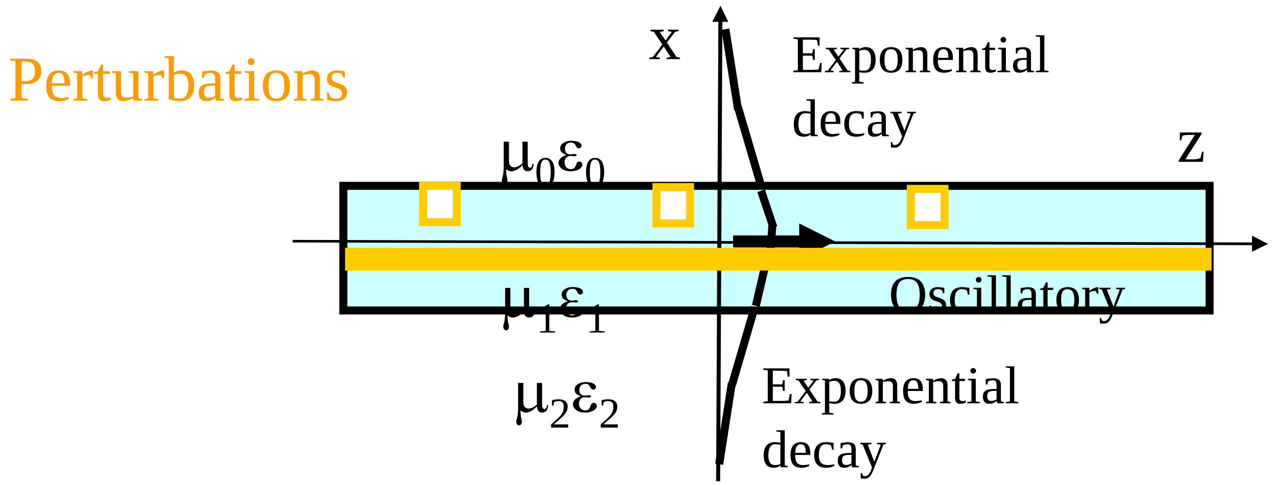
$$\nabla_t\left(\overline{E}_\nu \times \overline{H}_\mu^* + \overline{E}_\mu^* \times \overline{H}_\nu\right) - j\left(\beta_\nu - \beta_\mu\right)\left(\overline{E}_{t\nu} \times \overline{H}_{t\mu}^* + \overline{E}_{t\mu}^* \times \overline{H}_{t\nu}\right) = 0$$

- Apply divergence theorem to cross section, argue integral over contour at infinity is zero, and
- finally apply to mode in reverse direction and add
- Result the transverse E crossed Transverse H integrated over the cross section is zero when the propagation constant of the two modes differs.
- Apply to find mode amplitudes produced by E_{TAN} and H_{TAN} on a cross sectional plane

$$a_\nu = \iint_\infty dxdy\left(\overline{E}_t \times \overline{H}_\nu^* + \overline{E}_\nu^* \times \overline{H}_t\right)$$

$$b_\nu = \iint_\infty dxdy\left(\overline{E}_t \times \overline{H}_\nu^* - \overline{E}_\nu^* \times \overline{H}_t\right)$$

Coupled-Mode Concept



- Consider a geometry or material change for which there is an additional source of excitation with complex polarization amplitude P
- This polarization can be due to the E field from a strong mode hitting a region of missing or added dielectric.
- This polarization source then drives other modes.
- This sourcing of other modes can occur simultaneously among modes and is know as coupled modes.
- The distribution of the polarization can also be made periodic in distance along the guide to couple in our out planewaves.

Coupled Mode Formalism (Cont.)

$$\nabla\left(\overline{E}_1 \times \overline{H}_2^* + \overline{E}_2^* \times \overline{H}_1\right) = -j\omega \overline{P}_1 \cdot \overline{E}_2^* + j\omega P_2^* \cdot \overline{E}1$$

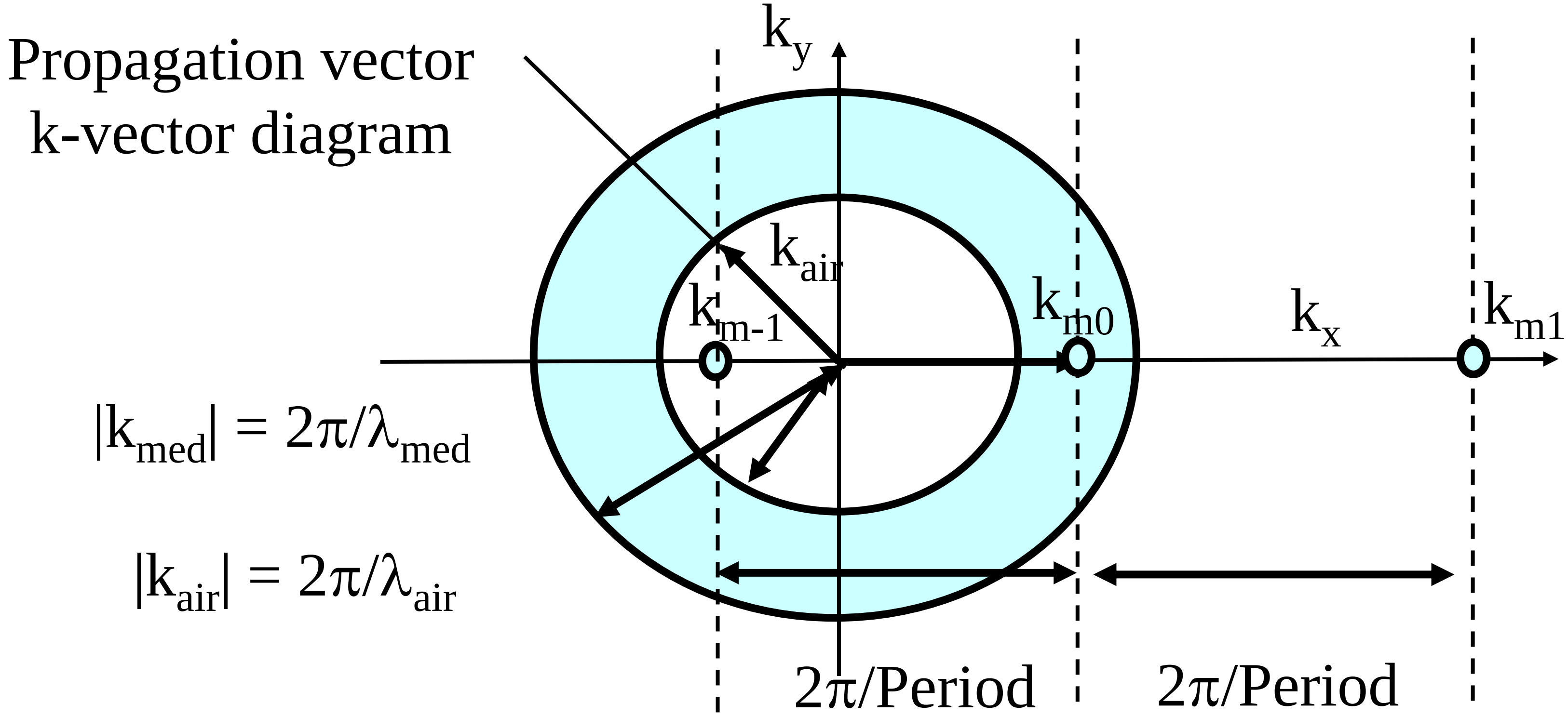
$$\overline{P} = \Delta\varepsilon\overline{E}$$

$$\overline{P}_i = \Delta\varepsilon_{ij}\overline{E}_j$$

$$\frac{da_\mu}{dz} + j\beta_u a_\mu = -j\omega\iint_\infty dxdy\overline{P}_{TOT} \cdot \overline{E}_\mu^*$$

$$\frac{db_\mu}{dz} - j\beta_u b_\mu = j\omega\iint_\infty dxdy\overline{P}_{TOT} \cdot \overline{E}_{-\mu}^*$$

Periodic Wave Vectors



- The mode k-vector is larger than k_O and smaller than k_G
- The periodic coupling creates new k-vectors spaced by 2π/Period
- The new k-vectors within the k_0 circle correspond to radiation waves
- Move upward vertically from k_{m-1} to find the k_y and angle.

Waveguide Deformations (Cont.)

Kogelnik 2.6

$$K_{vu}^{t} = \omega \iint_{\infty} dxdy \Delta\varepsilon \overline{E}_{tv} \cdot \overline{E}_{tu}^{*}$$

$$K_{vu}^{z} = \omega \iint_{\infty} dxdy \frac{\varepsilon}{\varepsilon + \Delta\varepsilon} \Delta\varepsilon \overline{E}_{zv} \cdot \overline{E}_{zu}^{*}$$

$$A_u' = -j \sum \left\{ \begin{array}{l} A_v \left(K_{vu}^{t} + K_{vu}^{z} \right) e^{-j(\beta_v - \beta_u) z} \\ + B_v \left(K_{vu}^{t} - K_{vu}^{z} \right) e^{j(\beta_v + \beta_u) z} \end{array} \right\}$$

$$B_u' = j \sum \left\{ \begin{array}{l} A_v \left(K_{vu}^{t} - K_{vu}^{z} \right) e^{-j(\beta_v + \beta_u) z} \\ + B_v \left(K_{vu}^{t} + K_{vu}^{z} \right) e^{j(\beta_v - \beta_u) z} \end{array} \right\}$$

- Substiture Pt and Pz contributions
- Introduce definitions of transverse and longitudinal K's and rewrite

Coupled-Wave Solutions: Co-Directional

Kogelnik 2.6.25-31

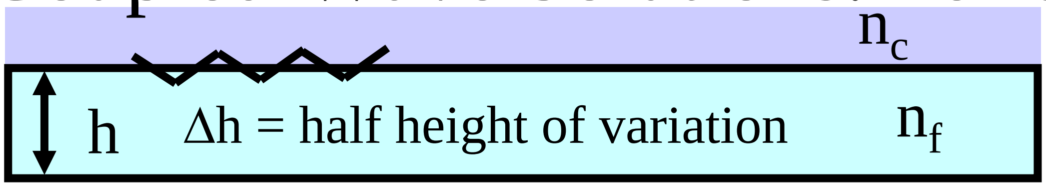
$$A' = -j\kappa B e^{-2j\delta z}$$
$$B' = -j\kappa A e^{2j\delta z}$$
$$A = \mathrm{R} e^{-j\delta z}$$
$$B = S e^{j\delta z}$$
$$R' - j\delta R = -j\kappa S$$
$$S' + j\delta S = -j\kappa R$$
$$R(0) = 1$$
$$S(0) = 0$$

- Select subset of terms
- Remove residual lack of synchronization
- Coupled Equations
- Matrix solution
- Boundary conditions

$$S(z) = -j\kappa \sin\left(z\sqrt{\kappa^2 + \delta^2}\right) / \sqrt{\kappa^2 + \delta^2}$$
$$R(z) = \cos\left(z\sqrt{\kappa^2 + \delta^2}\right) + j\delta \sin\left(z\sqrt{\kappa^2 + \delta^2}\right) / \sqrt{\kappa^2 + \delta^2}$$
$$S(z) = -j\kappa \sin(\kappa z)$$
$$R(z) = \cos(\kappa z)$$

- Simplification for synchronous case

Coupled-Wave Solutions: Periodic Waveguides



Kogelnik 2.6.41-48

$$h(z) = h_0 + \Delta h \cos(Kz)$$

$$K = 2\pi / \Lambda$$

$$\Delta\varepsilon = \varepsilon_0\left(n_f^2 - n_c^2\right)$$

$$\Delta\varepsilon = -\varepsilon_0\left(n_f^2 - n_c^2\right)$$

$$K_{u,-u}^1 = \omega \int_{-\infty}^{+\infty} dx \Delta\varepsilon E_y^2$$

$$K_{u,-u}^1 \approx \omega E_c^2 \int_{-\infty}^{+\infty} dx \Delta\varepsilon$$

$$K_{u,-u}^1 \approx \omega \varepsilon_0 E_c^2 \left(n_f^2 - n_c^2\right)\Delta h\left(e^{jKz} + e^{-jKz)}\right)$$

$$K_{u,-u}^1 \approx \frac{\pi}{\lambda}\frac{\Delta h}{h_{eff}}\frac{n_f^2 - N^2}{N}\left(e^{jKz} + e^{-jKz)}\right)$$

- Film n_f plus cover n_c
- Sinusoidal height
- Period Λ and k-vector K
- Two $\Delta\varepsilon$
- Ec is mode field at surface
- $\Delta\varepsilon$ z-variation produces (k-vector shift)
- N is the effective index

Coupled Modes as Eigenfunction Problem

Use to check Kogelnik Solution in Eq. 2.6.30-31.

$$\overline{X}_A = \begin{Bmatrix} a_{n-1} \\ a_n \\ a_{n+1} \end{Bmatrix}$$

$$\overline{X}'_A = \overline{\overline{M}} \cdot \overline{X}_A$$

$$\overline{X}'_{ei} = -j\lambda_i \overline{X}'_{ei}$$

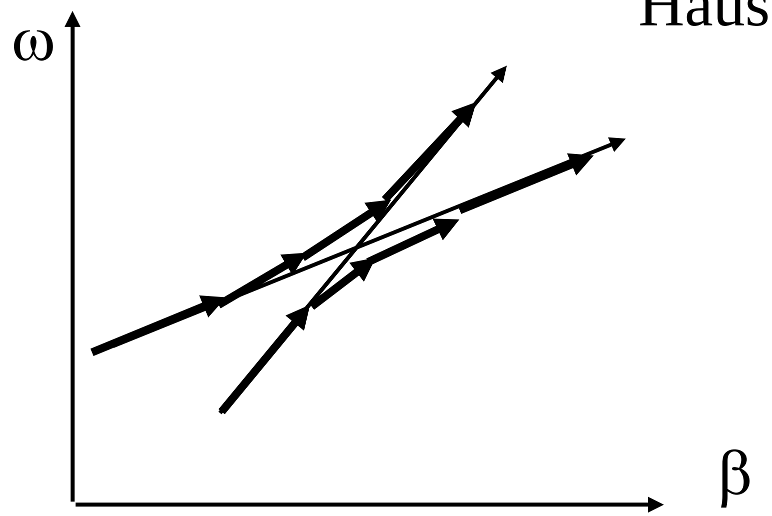
$$0 = \left[\overline{\overline{M}} - j\lambda_i \overline{\overline{I}}\right] \cdot \overline{X}_{ei}$$

$$Det\left[\overline{\overline{M}} - j\lambda_i \overline{\overline{I}}\right] = 0$$

- Construct a vector of mode amplitudes
- Rate equation can be written as derivative of mode vector equal to a coupling matrix M times mode vector
- Look for source free solutions (eigenvalues) by substituting an arbitrary exponential variation
- Determinant constrains arbitrary exponential (eigenvalues)

Coupled Mode: v_g and v_p Same Direction

Haus 7.6



When the group and phase velocities are in the same direction

- The eigenvalues (β's) move away from each other
- The displacement is proportional to the coupling coefficient
- The eigenfunctions (Super Modes) associated with eigenvalue (β) continuously change identity in passing through the crossing point

Radiating Zones

- Near (Static) Zone d << r << λ
  - Exponential is unity, => static and no radiation
- Intermediate (Induction) Zone d << r ~ λ
  - General expansion required
- Far (Radiation) Zone d << λ << r
  - Approximate denominator as 1/r
  - Approximate exponential as quadratic => Fresnel
  - Or Approximate exponential as linear => Fraunhoffer

$$|\bar{x} - \bar{x}' \approx r - \bar{n} \cdot \bar{x}'$$

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \bar{J}(\bar{x}') e^{ik(\bar{n} \cdot \bar{x}')} d^3 x'$$

Approximated by projection parallel to n In Fourier transform

Electric Dipole Fields and Radiation

$$\overline{A}(\overline{x}) = \frac{\mu_0}{4\pi}\int \overline{J}(\overline{x}')\frac{e^{ik|\overline{x}-\overline{x}'|}}{|\overline{x}-\overline{x}'|}d^3x' = -\frac{i\mu_0\omega}{4\pi}\,\overline{p}\,\frac{e^{ikr}}{r}$$

$$\overline{p} = \int \overline{x}'\rho(\overline{x}')d^3x'$$

$$\overline{H} \approx \frac{ck^2}{4\pi}(\overline{n}\times\overline{p})\frac{e^{ikr}}{r}$$

$$\overline{E} = Z_0\overline{H}\times\overline{n}$$

- Approximate exponent as constant
- Apply iωρ = Div J
- Integrate by parts
- Fields are perpendicular to n and perpendicular to each other
- Both E and H decrease as 1/r

Poynting Vector for Electric Dipole

$$\frac{dP}{d\Omega} = \frac{1}{2}\mathrm{Re}\left[r^2 \overline{n} \cdot \overline{E} \times \overline{H}^*\right]$$

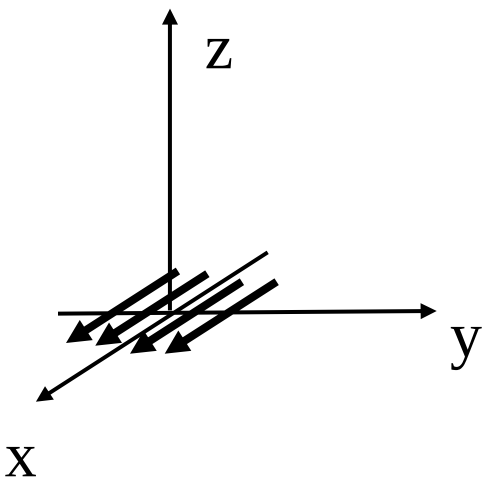
$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \left|(n \times p) \times n\right|^2$$

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 \left|\overline{p}^2\right| \sin^2\theta$$

$$P = \frac{c^2 Z_0}{12\pi} k^4 \left|\overline{p}^2\right|$$

- Poynting vector gives power density per unit solid angle
- Substitute for fields
- Sin squared polar angle
- Integrate over azimuthal and polar angles to get net power radiated.

Aperture Radiation



$$|\bar{x}-\bar{x}' \approx r-\bar{n}\cdot\bar{x}'$$

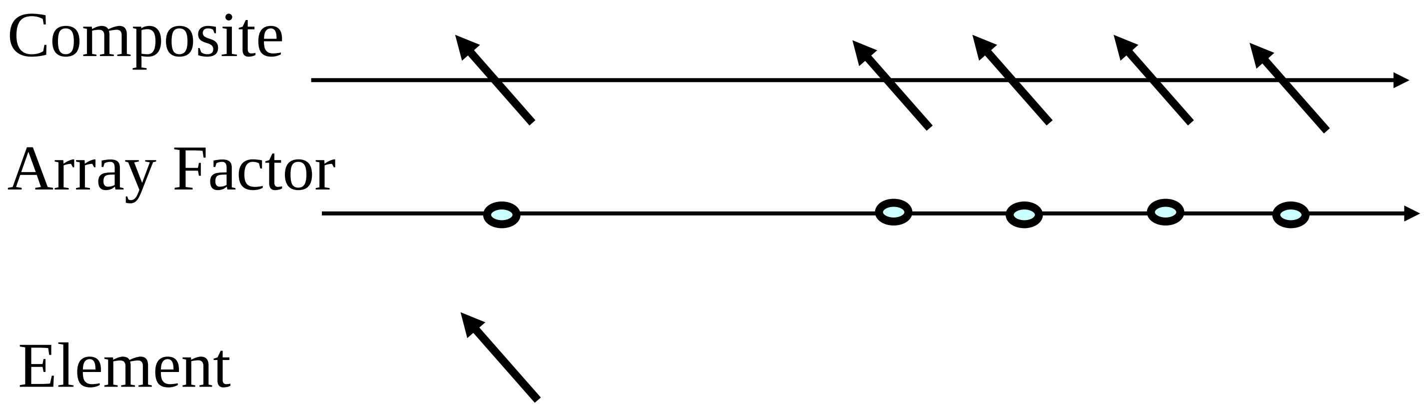
$$\bar{A}(\bar{x})=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \bar{J}(\bar{x}')e^{ik(\bar{n}\cdot\bar{x}')}d^3x'$$

$$\bar{A}(\bar{x})=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int_{-a/2}^{a/2}e^{ikx'}dx'\int_{-b/2}^{b/2}e^{iky'}dy'$$

$$\bar{A}(\bar{x})=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}ab\left[\frac{\sin(kax/2r)}{kax/2r}\frac{\sin(kay/2r)}{kay/2r}\right]$$

- Rectangular current patch flowing in x direction over
  - -a/2 < x < a/2
  - -b/2 < y < b/2
- Plug in Fraunhoffer approximation for A
- Factor to F(x)G(y)
- View as product of two Fourier Transforms

Antenna Array Patterns



- Composite Array id built from a element instantiated at array positions (convolution of element with space array factor)
- FT of convolution is product of FT's
- Composite pattern is the array pattern times element pattern.

Scattering by Dipoles Induced in Small Scatterers

Jackson 10.1.A

$$\overline{E}_{inc} = e_0 E_0 e^{ik\hat{n}_0 \cdot \overline{x}}$$

$$\overline{H}_{inc} = \hat{n}_0 \times \overline{E}_{inc} / Z_0$$

$$\overline{p} = induced\_electric\_dipole$$

$$\overline{m} = induced\_magnetic\_dipole$$

$$\overline{E}_{sc} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r} \left[(\hat{n} \times \overline{p}) \times n - n \times \overline{m} / c \right]$$

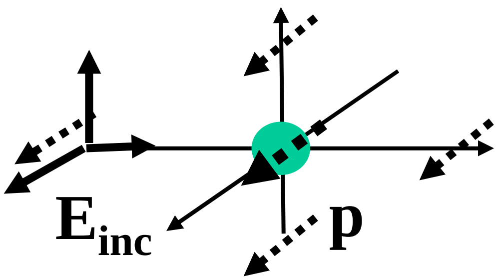
$$\overline{H}_{sc} = \hat{n} \times \overline{E}_{sc} / Z_0$$

- Incident fields induce electric and magnetic dipole moments
- Far fields from are then found from these moments

Scattering from a Small Dielectric Sphere

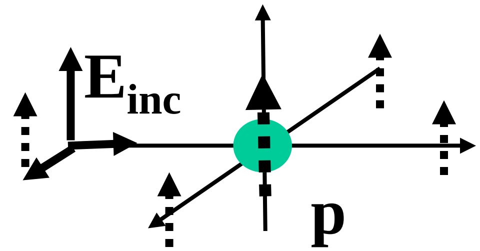
$$\bar{p} = 4\pi\varepsilon_0 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right) a^3 \bar{E}_{inc}$$

Jackson 10.1.B



$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{e}; \hat{n}_0, \hat{e}_0) = k^4 a^6 \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2 \left| \hat{e}^* \cdot \hat{e}_0 \right|^2$$

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3} k^4 a^6 \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2$$



- Dipole **p** is in the direction of the incident field and equal to the static polarization (same weight factor and proportional to volume).
- Radiation is proportional the observation polarization direction dotted with the incident polarization. This gives cosθ in one angle and constant in ϕ.
- Strength is 6-th power of size (volume squared) and 4-th power relative to size in wavelengths. (This explains the creation of the blue sky success of horizontally polarized sun glasses).
- Strongest and equal in forward and backward directions.

Scattering from a Small p.e.c. Sphere

$$\overline{p} = 4\pi\varepsilon_0 a^3 \overline{E}_{inc}$$

Jackson 10.1.C

$$\overline{m} = -2\pi a^3 \overline{H}_{inc}$$

$$\frac{d\sigma}{d\Omega}(\hat{n},\hat{e};\hat{n}_0,\hat{e}_0) = k^4 a^6 \left| \hat{e}^* \cdot \hat{e}_0 - \frac{1}{2}(\hat{n}\times\hat{e}^*)\cdot(\hat{n}_0\times\hat{e}_0) \right|^2$$

p and **m**

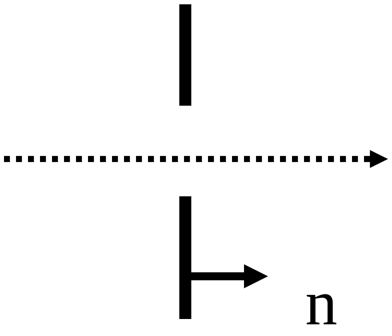
- Both exist
- Are at right angles
- Interfere coherently
  - produce a + b cosθ type patterns
  - low forward (1/3) and high backward (2x) scattering

Kirchhoff Approximation Representation

Jackson 10.5

$$\psi_{GEN}(\bar{x}) = -\frac{1}{4\pi}\oint_{S_1} \frac{e^{ikR}}{R}\bar{n}' \cdot \left[\nabla'\psi + ik\left(1+\frac{i}{kR}\right)\frac{\bar{R}}{R}\psi\right]da'$$

$$\psi_D(\bar{x}) = -\frac{1}{2\pi i}\oint_{S_1} \frac{e^{ikR}}{R}\left(1+\frac{i}{kR}\right)\frac{\bar{n}'\cdot\bar{R}}{R}\psi(\bar{x}')da'$$



- Apply to Screen with aperture
- Assumptions
  - ψ and its normal derivative vanish except on opening
  - ψ and its derivative are equal to the those incident on aperture with no screen
- Inherent inconsistencies
  - Since scattered field is zero everywhere on screen it is zero everywhere
  - Integral does not yield the assumed values on the openings
- Enforcing either Dirichlet or Neuman Boundary Conditions results in a consistent formulation

Vector Integral Representation for Far Field

Jackson 10.7

$$E(\bar{x}) = \oint_S \left[\bar{E}(\bar{n}' \cdot \nabla' G) - G(\bar{n}' \cdot \nabla')\bar{E}\right] da'$$

$$E(\bar{x}) = \oint_S \left[i\omega(\bar{n}' \times \bar{B})G + (\bar{n}' \times \bar{E}) \times \nabla' G + (\bar{n}' \cdot \bar{E})\nabla' G\right] da'$$

$$G \to \frac{e^{ikr'}}{4\pi r'} e^{ik\hat{n}' \cdot \bar{x}}$$

$$\bar{E}_s'(\bar{x}) \to \frac{e^{ikr}}{r} \bar{F}(\bar{k}, \bar{k}_0)$$

$$\hat{e}^* \cdot \bar{F}(\bar{k}, \bar{k}_0) = \frac{i}{4\pi} \oint_{S_1} e^{i\bar{k} \cdot \bar{x}} \left[\omega \hat{e}^* \cdot (\bar{n}' \times \bar{B}_s) + \hat{e}^* \cdot \left(\bar{k} \times (\bar{n}' \times \bar{E}_s)\right)\right] da'$$

- Start with **x** in volume and interaction integral
- Treat **x** as singular point plus rest of volume
- Apply divergence theorem
- Use free space Green Function
- Integral on surface at infinity goes to zero
- Rewrite in **transvere only** components of E and B on surface

Diffraction by a Circular Aperture Far Field

$$\overline{E}(\overline{x}) = \frac{ie^{ikr}E_0\cos\alpha}{2\pi r}\int_0^a \rho d\rho \int_0^{2\pi} d\beta e^{ik\rho[\sin\alpha\cos\beta - \sin\theta\cos(\phi-\beta)]}$$

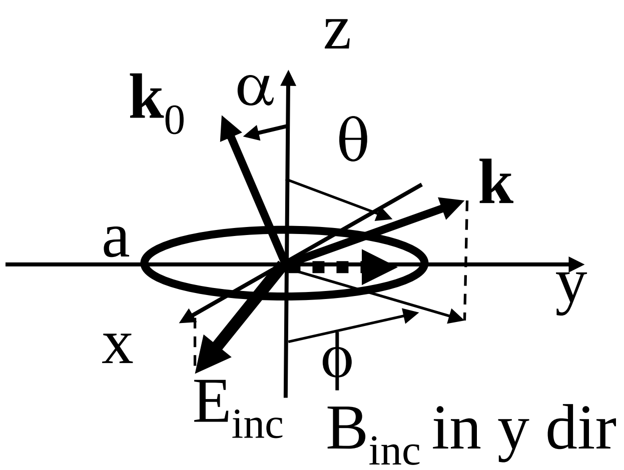
$$\xi = \left(\sin^2\theta + \sin^2\alpha - 2\sin\theta\sin\alpha\cos\phi\right)^{1/2}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\beta' e^{-ik\rho\xi\cos\beta'} = J_0(k\rho\xi)$$

$$\overline{E}(\overline{x}) = \frac{ie^{ikr}}{r}a^2E_0\cos\alpha\left(\overline{k}\times\overline{e}_2\right)\frac{J_1(ka\xi)}{ka\xi}$$

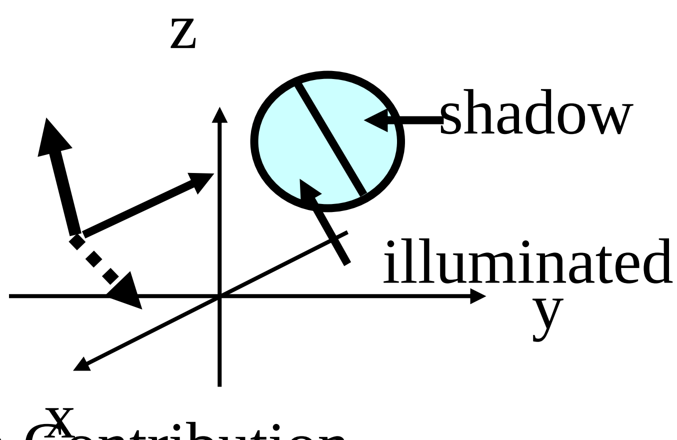
$$\frac{dP}{d\Omega} = P_i\cos\alpha\frac{(ka)^2}{4\pi}\left(\cos^2\theta + \cos^2\phi\sin^2\theta\right)\left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2$$

$$P_i = \left(\frac{\overline{E}_0^2}{2Z_0}\right)\pi a^2\cos\alpha$$



- Plane wave in x-z plane incident from below
  - E_{TAN} reduced by cos α; linear phase in x direction
- Find field in direction k
  - linear phase in x and y directions
  - Combine all phases; recognize azimuthal integral as J_0; integrate in ρ => J_1
- Result is $J_1(v)/v$ with weighting for tangential components of arrival and scattering

Scattering in the Short Wavelength Limit



- Shadowed Region Contribution
  - Boundary Condition $E_s = -E_{inc}$; $B_s = -B_{inc}$
  - Small Ave except forward => depend only on projected area (diffraction pattern from the shadow)
- Illuminated Region Contribution
  - Boundary Conditions $E_s = -E_{inc}$; $B_s = -B_{inc}$ SAME as Ill.!!!
  - Normal difference gives sign difference and different result
  - Stationary phase brings our specular surface contributions
- Shadow diffraction can dominate in forward direction
  - See Figure 10.16

Planewave Expansion

$$\overline{E}_s'(\overline{x}) \rightarrow \frac{e^{ikr}}{r}\overline{F}(\overline{k},\overline{k}_0)$$

Jackson 10.7

$$\hat{e}^* \cdot \overline{F}(\overline{k},\overline{k}_0) = \frac{i}{4\pi}\oint_{S_1} e^{i\overline{k}\cdot\overline{x}}\left[\omega\hat{e}^* \cdot (\overline{n}' \times \overline{B}_s) + \hat{e}^* \cdot \left(\overline{k} \times \left(\overline{n}' \times \overline{E}_s\right)\right)\right] da'$$

Example for a mask with period P in x direction.

$$\overline{E}_{TOTAL} = \sum_n \overline{E}_n e^{-j(k_0 \sin(\theta_{no})x_o + k_0 \cos(\theta no)z_o)}$$

$$\overline{E}_{TOTAL} = \sum_{n=-N}^{N} \overline{E}_n A(\theta_{ni}) e^{j\Phi(\theta_{ni})} e^{-j(k_0 \sin(\theta_{ni})x_i + k_0 \cos(\theta ni)z_i)}$$

- Start from **Transverse components** of **E** and **B** on plane
- Make planewave spectrum expansion between mask and lens (assume periodic and switch to e^{jwt})
- Lens then low pass filters and apodizes/phases transmitted spectrum
- Propagation to image plane is thus the Fourier Transform of the filtered/phased spectrum

Electric Field as Sum of Plane Waves

Simplify to (x,z) plane, E in y-direction, A = 1, Φ =0

Mask with period P Bragg Condition Implies

$$\sin(\theta_n) = \frac{n\lambda}{P} \qquad \cos(\theta_n) = \sqrt{1-\left(\frac{n\lambda}{P}\right)^2}$$

$$\overline{k_n} = k_{x_n}\hat{x} + k_{z\,n}\hat{z} = k_0 \sin\left(\theta_{x_n}\right)\hat{x} + k_0 \cos\left(\theta_{z_n}\right)\hat{z}$$

$$E_{TOTAL} = \sum_n E_n e^{-j(k_0 \sin(\theta_n)x + k_0 \cos(\theta n)z)} = \sum_n E_n e^{-j(\overline{k}_n \cdot \overline{x})}$$

Three wave case for on-axis illumination of mask with period P

$$E_{TOTAL} = E_{-1} e^{-j(-\frac{2\pi}{P}x + \frac{2\pi}{\lambda}\cos(\theta_{-1})z)} + E_0 e^{-j(0\cdot\frac{2\pi}{P}x + \frac{2\pi}{\lambda}\cos(\theta_0)z)} + E_{+1} e^{-j(\frac{2\pi}{P}x + \frac{2\pi}{\lambda}\cos(\theta_{+1})z)}$$

Electric Field Spectrum M(u) from E(x)

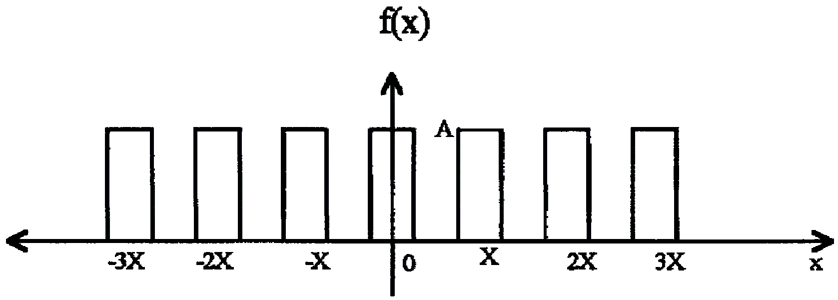
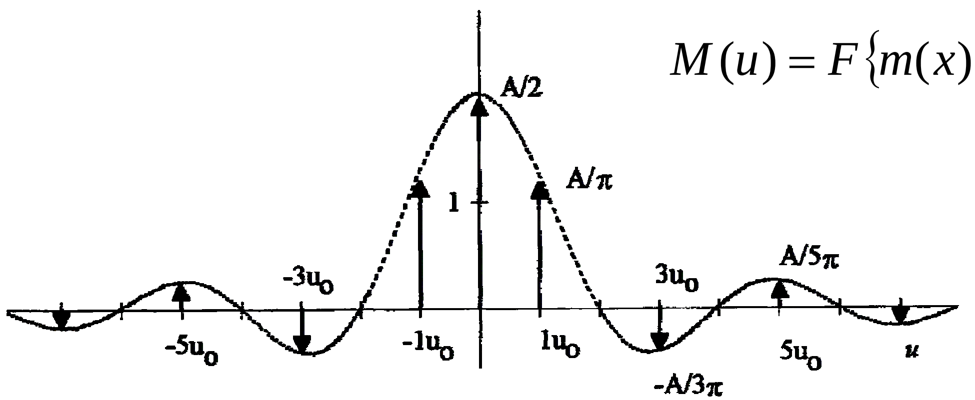


Figure 18 A periodic rectangular wave, representing dense mask features.

$$m(x) = A \cdot rect\left(\frac{x}{P/2}\right) * comb\left(\frac{x}{P}\right)$$

$$M(u) = F\{m(x)\} = F\left\{A \cdot rect\left(\frac{x}{P/2}\right)\right\} \cdot F\left\{comb\left(\frac{x}{P}\right)\right\}$$



$$M(u) = \frac{A}{2}\sin c\left(\frac{u}{2u_0}\right)\sum_{n=-\infty}^{\infty}\delta(u-u_0)$$

Figure 19 The amplitude spectrum of a rectangular wave, A/2 sinc($u/2u_0$). This is equivalent to the discrete orders of the coherent Fraunhofer diffraction pattern.

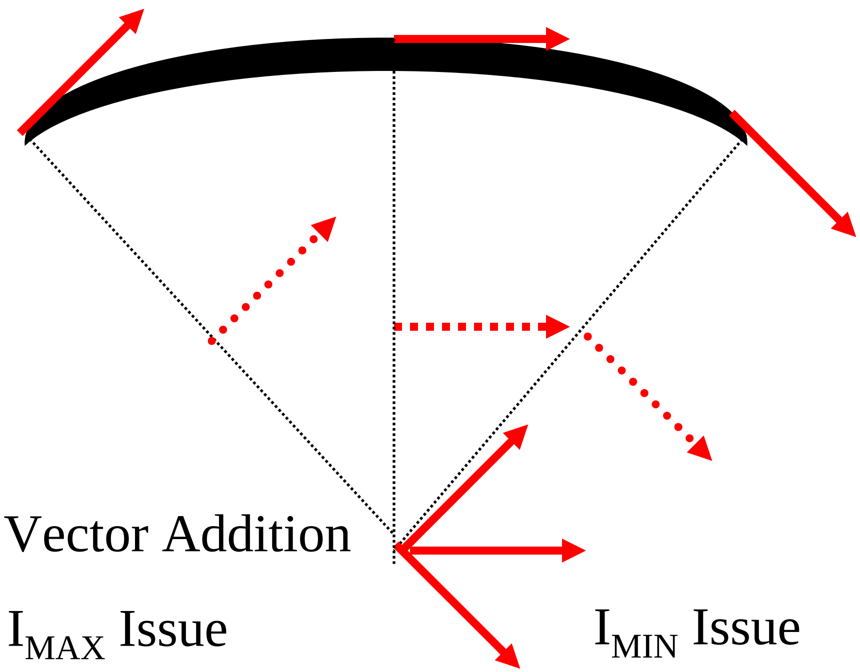
u_0 = 1/P

Values are ½, 1/π, 1/3π, 1/5π

Sheats and Smith

Polarization Effects at High NA

Parallel Orientation



Vector Addition

I_{MAX} Issue

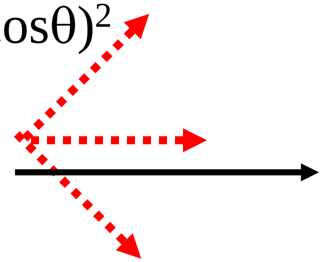
$E_X \sim \cos\theta$

$I_{MAX} \sim (\cos\theta)^2$

I_{MIN} Issue

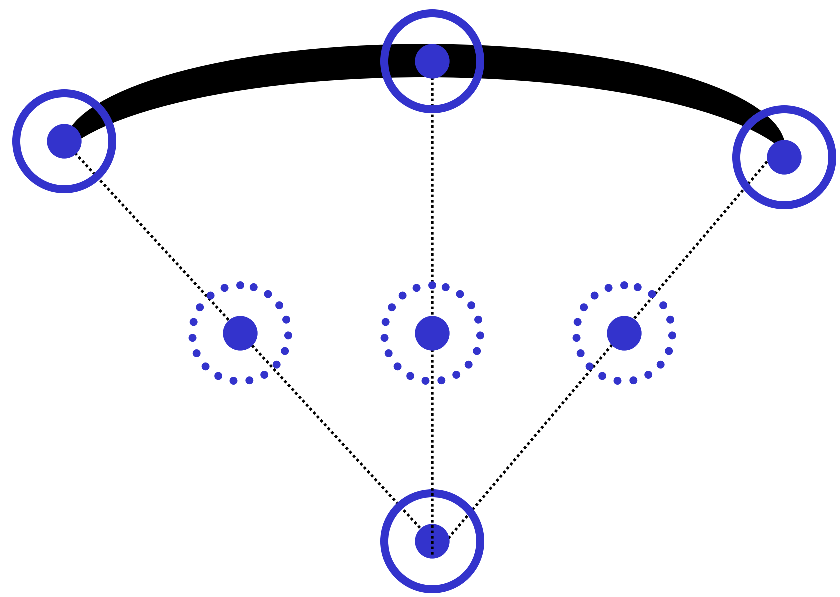
$E_Z \sim \sin\theta$

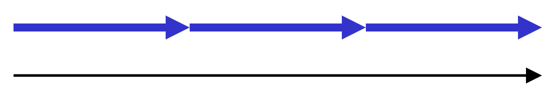
$I_{MIN} \sim (\sin\theta)^2$



$\vec{E}_{total}$

Perpendicular Orientation





$\vec{E}_{total}$

Alternating Phase-Shifting Mask

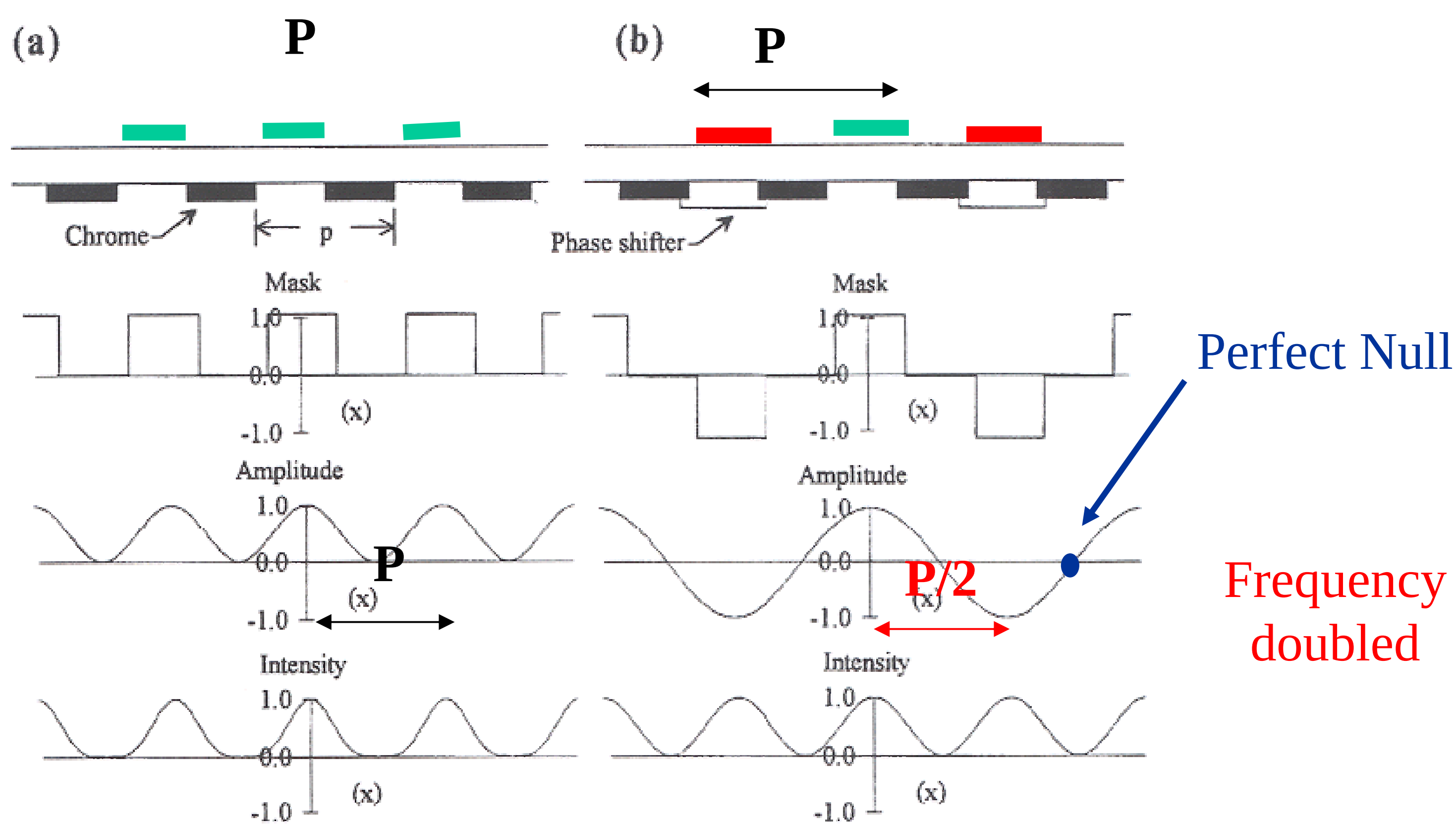
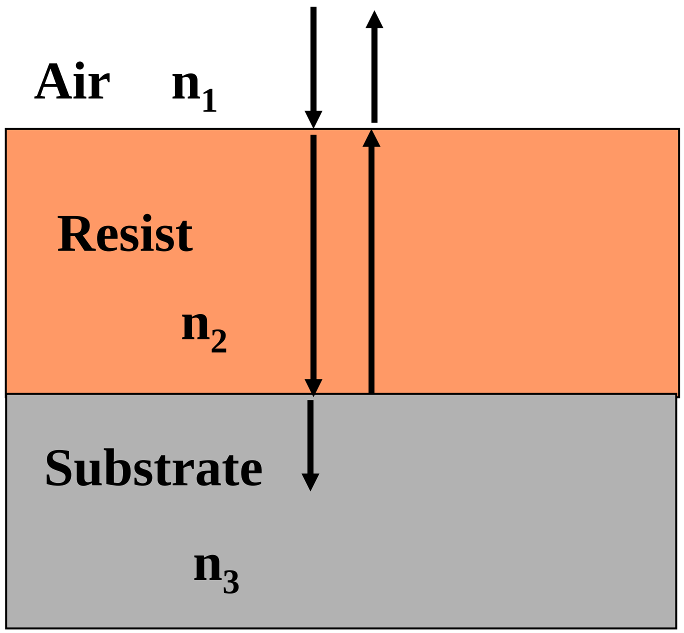


Figure 64 Schematic of (a), a conventional binary mask (b) an alternating phase shift mask. The mask electric field, image amplitude, and image intensity is shown for each.

Sheats and Smith

Electric Field within Resist



5 waves
match boundary conditions
(or use signal flow analysis)
use definition of τ_D

$$E_{RESIST}(x,y,z) = E_{AIR\_INC}(x,y)\frac{\tau_{12}\left(e^{-jk_2 z} + \rho_{23}\tau_D^2 e^{+jk_2 z}\right)}{1+\rho_{12}\rho_{23}\tau_D^2}$$

Downward wave · **Round trip propagation** · **Upward wave** · **Transmission in** · **Reflection at substrate** · **Round trip loop gain (loss)**

$$\rho_{12} = -\rho_{21}$$

Reflection and Transmission

Reflection and transmission coefficients in going from media i to media j

$$\rho_{ij} = \frac{n_i - n_j}{n_i + n_j} \qquad \tau_{ij} = \frac{2n_i}{n_i + n_j}$$

Note: 1 + ρ = τ

Phase change and attenuation with distance z

Example: air to quartz (n_{qz} = 1.5); ρ = -0.2 and τ = 0.8

$$\tau(z) = e^{-j(n_r + jn_i)\frac{2\pi}{\lambda_{air}}z}$$

Example: complex propagation factor in going from z=0 to z=D is

$$\tau_D = e^{-j(n_r + jn_i)\frac{2\pi}{\lambda_{air}}D}$$

The same net complex factor occurs for the upward wave in going from z=D to z=0.