EE243 Advanced Electromagnetic Theory

Lec # 23 Scattering and Diffraction

- **Scalar Diffraction Theory**
- **Vector Diffraction Theory**
- **Babinet and Other Principles**
- **Optical Theorem**

Reading: Jackson Chapter 10.5-10.9, 10.10-10.11 lite

Overview

Objects large compared to a wavelength are generally treated by approximate integrals over the assumed fields on their surfaces.

- In many cases (where the polarization is not important) scalar diffraction can be used.
- Where polarization effects are important a vector formulation is needed.
- The two key factors in the approximation
  - The assumed fields on the surfaces or apertures
  - The source free Green’s function used in the integral

Scalar Integral Representation for Far Field

$$\psi(\bar{x}) = \oint_S \left[\psi(\bar{x}')\bar{n}' \cdot \nabla' G(\bar{x}, \bar{x}') - G(\bar{x}, \bar{x}')\bar{n}' \cdot \nabla' \psi(\bar{x}')\right] da'$$

Jackson 10.5

$$G(\bar{x}, \bar{x}') = \frac{e^{ikR}}{4\pi R}$$

$$\bar{R} = \bar{x} - \bar{x}'$$

$$\psi(\bar{x}) = -\frac{1}{4\pi} \oint_S \frac{e^{ikR}}{4\pi R} \bar{n}' \cdot \left[\nabla' \psi + ik\left(1 + \frac{i}{kR}\right)\frac{\bar{R}}{R}\psi\right] da'$$

$$\psi \to f(\theta, \phi)\frac{e^{ikr}}{4\pi r} \Rightarrow \frac{1}{\psi}\frac{\partial \psi}{\partial r} \to \left(ik - \frac{1}{r}\right)$$

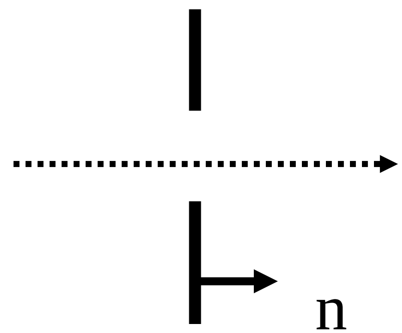
- General Representation for solution to scalar wave equation
- Choose scalar Green’s function (**R** to simplifies notation)
- Integral that closes surface at infinity goes to zero
  - radiation condition
  - f(θ,ϕ) is the radiation pattern

Kirchhoff Approximation Representation

Jackson 10.5

$$\psi_{GEN}(\bar{x}) = -\frac{1}{4\pi}\oint_{S_1} \frac{e^{ikR}}{R}\,\bar{n}' \cdot \left[\nabla'\psi + ik\left(1+\frac{i}{kR}\right)\frac{\bar{R}}{R}\psi\right] da'$$

$$\psi_D(\bar{x}) = -\frac{1}{2\pi i}\oint_{S_1} \frac{e^{ikR}}{R}\left(1+\frac{i}{kR}\right)\frac{\bar{n}'\cdot\bar{R}}{R}\,\psi(\bar{x}')\,da'$$



- Apply to Screen with aperture
- Assumptions
  - ψ and its normal derivative vanish except on opening
  - ψ and its derivative are equal to the those incident on aperture with no screen
- Inherent inconsistencies
  - Since scattered field is zero everywhere on screen it is zero everywhere
  - Integral does not yield the assumed values on the openings
- Enforcing either Dirichlet or Neuman Boundary Conditions results in a consistent formulation

Kirchhoff Approximation: Green's Function

Jackson 10.5

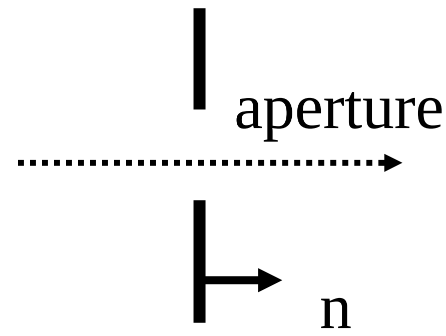
$$\psi_{GEN}(\overline{x}) = -\frac{1}{4\pi}\oint_{S_1} \frac{e^{ikR}}{R}\,\overline{n}' \cdot \left[\nabla'\psi + ik\left(1+\frac{i}{kR}\right)\frac{\overline{R}}{R}\psi\right] da'$$

$$\psi(\overline{x}) = -\frac{k}{2\pi i}\oint_{S_1} \frac{e^{ikr}}{r}\frac{e^{ikr'}}{r'}\vartheta(\theta,\theta')\,da'$$

$$\vartheta(\theta,\theta') = \cos\theta$$

$$\vartheta(\theta,\theta') = \cos\theta'$$

$$\vartheta(\theta,\theta') = \frac{1}{2}(\cos\theta + \cos\theta')$$



Example for a point source on one side of Screen

- Approximating ψ, δψ/δn or keeping both (Kirchhoff) gives the same integral except for the **obliquity factor** ϑ(θ,θ') that weights the rays by the cos of the arrival or takeoff angle.

Vector Integral Representation for Far Field

Jackson 10.7

$$E(\bar{x}) = \oint_S \left[\bar{E}(\bar{n}' \cdot \nabla' G) - G(\bar{n}' \cdot \nabla')\bar{E} \right] da'$$

$$E(\bar{x}) = \oint_S \left[i\omega(\bar{n}' \times \bar{B})G + (\bar{n}' \times \bar{E}) \times \nabla' G + (\bar{n}' \cdot \bar{E})\nabla' G \right] da'$$

$$G \rightarrow \frac{e^{ikr'}}{4\pi r'} e^{ik\hat{n}' \cdot \bar{x}}$$

$$\bar{E}'_s(\bar{x}) \rightarrow \frac{e^{ikr}}{r} \bar{F}(\bar{k}, \bar{k}_0)$$

$$\hat{e}^* \cdot \bar{F}(\bar{k}, \bar{k}_0) = \frac{i}{4\pi} \oint_{S_1} e^{i\bar{k} \cdot \bar{x}} \left[\omega \hat{e}^* \cdot (\bar{n}' \times \bar{B}_s) + \hat{e}^* \cdot \left(\bar{k} \times (\bar{n}' \times \bar{E}_s) \right) \right] da'$$

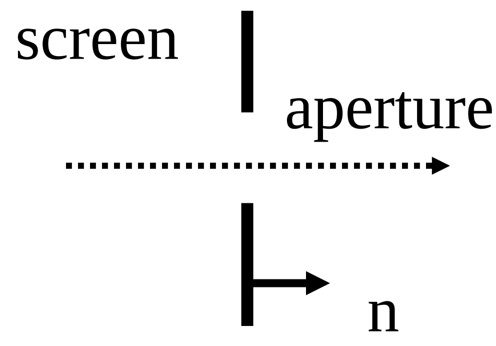
- Start with **x** in volume and interaction integral
- Treat **x** as singular point plus rest of volume
- Apply divergence theorem
- Use free space Green Function
- Integral on surface at infinity goes to zero
- Rewrite in **transvere only** components of E and B on surface

Diffraction by Screen with Aperture

$$\overline{A}(\overline{x}) = \frac{1}{2\pi} \int_{screen} (\hat{n} \times \overline{B}) \frac{e^{ikR}}{R} da'$$

$$B'(\overline{x}) = \frac{1}{2\pi} \nabla \times \int_{screen} (\hat{n} \times \overline{B}) \frac{e^{ikR}}{R} da'$$

$$\overline{E}_{diff}(\overline{x}) = \frac{1}{2\pi} \nabla \times \int_{apertures} (\hat{n} \times \overline{E}) \frac{e^{ikR}}{R} da'$$



- B is given by integrating B values on the screen geometry.
- E is given by integrating E fields on the apertures
- This suggest that dual problems are related (Babinet’s principle)

Vector Theorems and Concepts

Jackson 10.8

- Equivalence theorem: Contributions from sources outside of a volume can be found from tangential **E** and **H** on the surface of the volume.
- Reaction integral: Integral of tangential fields on the surface is same as calculating E dot J and H dot M throughout the volume.
- Green's Function choice: The region outside the volume could be filled with p.e.c. material to cancel **E** and double effect of **H** or magnetic material to cancel **H** and double the effect of **E**
- Babinet's Principle: For perfectly conducting thin screen and its complement the electric and magnetic fields for complementary problems are given by the same integral. For example in the case of a slot the magnetic field in a an aperture is used and the complementary case is a metallic bar (screen) and the electric field over the bar is used.

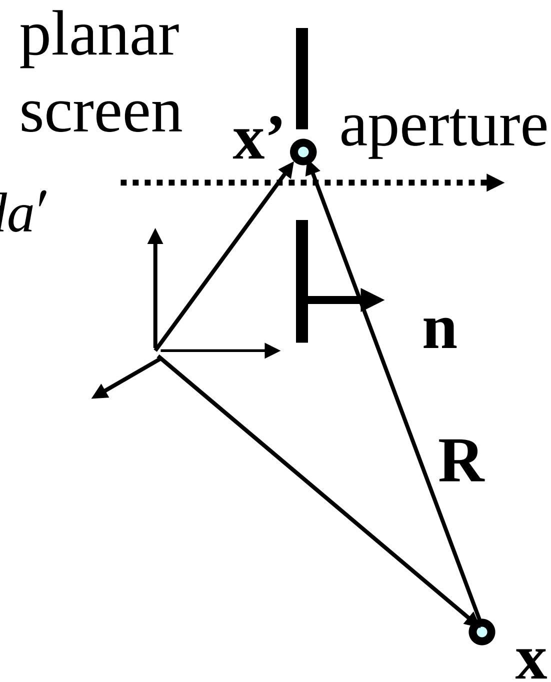
Diffraction by a Circular Aperture Far Field

Jackson 10.9

$$kR = kR - k\hat{n}\cdot\bar{x}' + \frac{k}{2r}\left[r'^2 - \left(\hat{n}\cdot\bar{x}'\right)^2\right] + \dots$$

$$\psi_{SCALAR}(\bar{x}) = -\frac{e^{ikr}}{4\pi r}\int_{S_1} e^{-i\bar{k}\cdot\bar{x}'}\left[\hat{n}\cdot\nabla'\psi\left(\bar{x}'\right) + i\bar{k}\cdot n\,\psi\left(\bar{x}'\right)\right]da'$$

$$\bar{E}_{VECTOR}\left(\bar{x}\right) = \frac{ie^{ikr}}{2\pi r}\bar{k}\times\int_{S_1}\left(\hat{n}\times\bar{E}\left(\bar{x}'\right)\right)e^{i\bar{k}\cdot\bar{x}'}da'$$



- Approximate kR by Taylor series
- Use Scalar or
- Use Vector for A plus B curl A, E curl B

Diffraction by a Circular Aperture Far Field

$$\overline{E}(\overline{x}) = \frac{ie^{ikr} E_0 \cos\alpha}{2\pi r} \int_0^a \rho d\rho \int_0^{2\pi} d\beta e^{ik\rho[\sin\alpha\cos\beta - \sin\theta\cos(\phi-\beta)]}$$

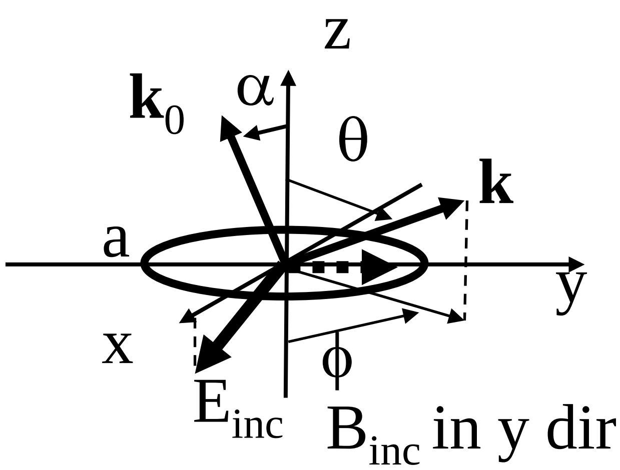
$$\xi = \left(\sin^2\theta + \sin^2\alpha - 2\sin\theta\sin\alpha\cos\phi\right)^{1/2}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\beta' e^{-ik\rho\xi\cos\beta'} = J_0(k\rho\xi)$$

$$\overline{E}(\overline{x}) = \frac{ie^{ikr}}{r} a^2 E_0 \cos\alpha\left(\overline{k}\times\overline{e}_2\right)\frac{J_1(ka\xi)}{ka\xi}$$

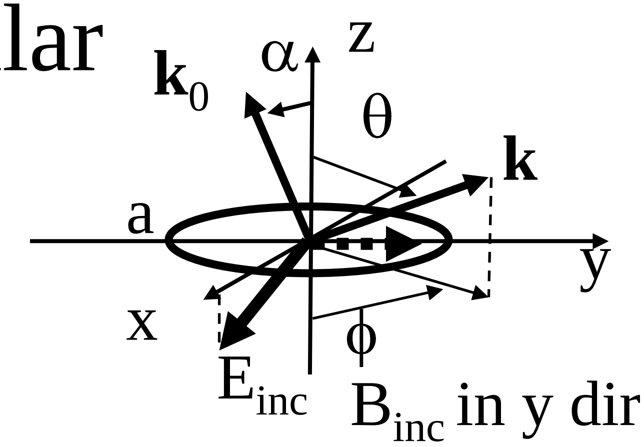
$$\frac{dP}{d\Omega} = P_i \cos\alpha \frac{(ka)^2}{4\pi}\left(\cos^2\theta + \cos^2\phi\sin^2\theta\right)\left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2$$

$$P_i = \left(\frac{\overline{E}_0^2}{2Z_0}\right)\pi a^2\cos\alpha$$



- Plane wave in x-z plane incident from below
  - E_{TAN} reduced by cos α; linear phase in x direction
- Find field in direction k
  - linear phase in x and y directions
  - Combine all phases; recognize azimuthal integral as J_0; integrate in ρ => J_1
- Result is $J_1(v)/v$ with weighting for tangential components of arrival and scattering

Diffraction by a Circular Aperture Far Field: Vector versus Scalar



$$\bar{E}(\bar{x}) = \frac{ie^{ikr}}{r} a^2 E_0 \cos\alpha\left(\bar{k}\times\bar{e}_2\right)\frac{J_1(ka\xi)}{ka\xi}$$

$$\frac{dP}{d\Omega} = P_i \cos\alpha \frac{(ka)^2}{4\pi}\left(\cos^2\theta + \cos^2\phi\sin^2\theta\right)\left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2$$

$$P_i = \left(\frac{\bar{E}_0^2}{2Z_0}\right)\pi a^2 \cos\alpha$$

$$\left|\bar{E}\right| = \psi(\bar{x}) = ik\frac{e^{ikr}}{r} a^2 E_0\left(\frac{\cos\alpha + \cos\theta}{2}\right)\frac{J_1(ka\xi)}{ka\xi}$$

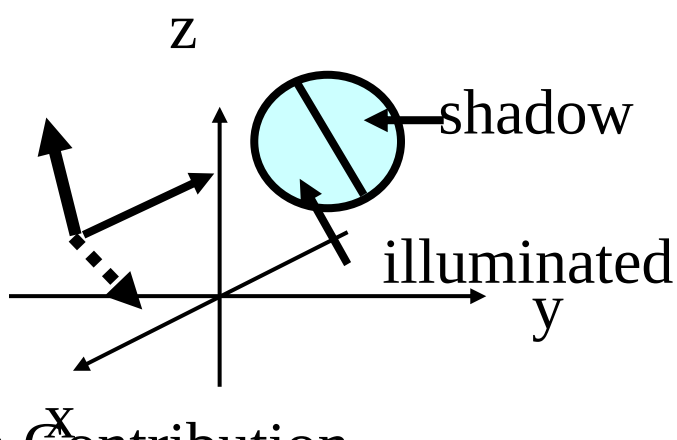
$$\frac{dP}{d\Omega} \approx P_i \frac{(ka)^2}{4\pi}\cos\alpha\left(\frac{\cos\alpha+\cos\theta}{2\cos\alpha}\right)^2\left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2$$

$$\alpha = 0$$

$$\frac{dP}{d\Omega} \approx P_i \frac{(ka)^2}{\pi}\left|\frac{J_1(ka\xi)}{ka\xi}\right|^2$$

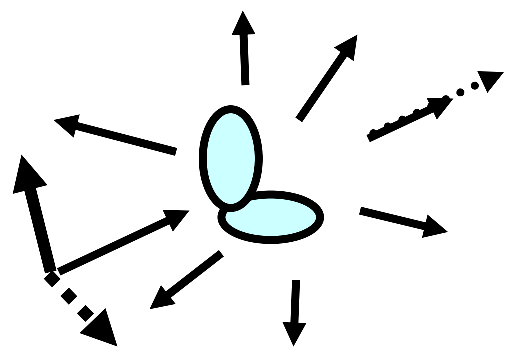
- Compare Scalor Kirchhoff
- Vector
- Difference: Obliquity type factors
- Agree when α = 0

Scattering in the Short Wavelength Limit



- Shadowed Region Contribution
  - Boundary Condition $E_s = -E_{inc}$; $B_s = -B_{inc}$
  - Small Ave except forward => depend only on projected area (diffraction pattern from the shadow)
- Illuminated Region Contribution
  - Boundary Conditions $E_s = -E_{inc}$; $B_s = -B_{inc}$ SAME as Ill.!!!
  - Normal difference gives sign difference and different result
  - Stationary phase brings our specular surface contributions
- Shadow diffraction can dominate in forward direction
  - See Figure 10.16

Optical Theorem



Reduction in forward direction proportional to the total power radiated

- Know Far Field Expression
- Look at Near Field Poynting Vector to determine the total power taken from the wave
- With substitution, some manipulation the integral pf the total power becomes proportional to the integral for the scattering amplitude in the forward direction.
- Physical interpretation: The total power taken from the wave must appear in a commensurate reduction in the total field in the forward direction.
- How can an electric field be related to Power? (as a perturbation)
  - $E_{tot} = E_{inc} + E_y$ and $E_y << E_{inc}$
  - the power EE* $\sim E_{inc}E_{inc}^*(1-2E_s/E_{inc})$
  - Power reduction is proportional to E_s

Dielectric Properties and Crystal Opalesence

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{Ne^2}{\varepsilon_0 m} \sum_j \frac{f_j}{\left(\omega_j^2 - i\omega\gamma_j - w\omega^2\right)}$$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{4\pi N}{k^2} e_0^* \cdot f\left(\bar{k} = \bar{k}_0\right)$$

$$\bar{p} = \frac{e^2}{m} \sum_j \frac{f_j}{\left(\omega_j^2 - i\omega\gamma_j - w\omega^2\right)}$$

$$e_0^* \cdot f\left(\bar{k} = \bar{k}_0\right) = \frac{e^2 k^2}{4\pi\varepsilon_0 m} \sum_j \frac{f_j}{\left(\omega_j^2 - i\omega\gamma_j - w\omega^2\right)}$$

Reduction in forward direction proportional to the total power radiated

- Media Model Chapter 7.3.A (Harmonic Oscillator)
- Relate to Optical Theorem 10.11 (evaluate absorption)
- Include mechanical interaction from force on atoms 10.3.D
- Crystal Opalesence occurs when the forces between atoms cause mechanical linkage of energy between atoms that produce density variations that affect EM absorption and scattering.