EE243 Advanced Electromagnetic Theory

Lec # 12: Waves in Dispersive Media

- **Models for Medial (harmonic oscillator)**
- **Behavior of permittivity and refractive index vs** ω
- **Superposition of waves and group velocity**
- **Pulse width increase with propagation**
- **Implications of Causality and initial/final condition**
- **Kramers Kronig Relations**
- $\omega-\beta$ **diagrams**

Reading: Jackson Ch 7.8-7.10 (skip 7.6 and 7.7)

Overview

- A harmonic oscillator model of electrons circulating about the nucleus is the basic phenomena by which materials affect EM waves.
- These ponderable media and also boundary (eigenvalue) constraints produce wave phase velocities that depend on frequency.
- This so called dispersion generally makes
  - the energy velocity less than the speed of light and (even undefined)
  - And causes the pulse length to increase with distance.
- $\varepsilon(\omega)/\varepsilon_0$ is an analytical function and the real part can be found from the imaginary part and visa versa
- The tool for characterizing wave dispersion is the ω-β diagram that plots the radian frequency versus the propagation k-vector.

Harmonic Oscillator Model for Material

$$m\left[\ddot{x} + \gamma\dot{x} + \omega_0^2\right] = -e\overline{E}\left(\overline{x}, t\right)$$

$$\overline{p} = -e\overline{x} = \frac{e^2}{m}\frac{E}{(\omega_0^2 - i\omega\gamma - \omega^2)}$$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{Ne^2}{\varepsilon_0 m}\sum_j f_j \frac{1}{(\omega_i^2 - i\omega\gamma_i - \omega^2)}$$

- 2nd order differential equation in time
- $e^{i\omega t}$ converts to 2nd order algebraic equation
- Polarization is charge times displacement
- Dielectric constant is 1 + polarization effects
- Add contributions of each oscillator type

Permittivity Frequency Behavior

- $\varepsilon(\omega)$ has frequency dependence
- Re $\varepsilon(\omega)$ generally decreases with increasing frequency
- Imy $\varepsilon(\omega)$ shows resonate peaks
- With low damping see resonate absorption and anomolous dispersion [negative slope of $\varepsilon(\omega)$]
- Represent $\varepsilon(\omega)$ by sum of poles in complex ω plane
- Free electrons give pole at zero (Drude Model)
- At high frequency converges to 1 as constant/ω^2

Refractive Index Water Versus Frequency

- Real part drops from 9 to 1.5 and then 1.0
- Imy Part rises to peak, drops to valley (visible), rises to peak, and drops to valley.
- First peak is due to vibrational modes in molecules and interaction among molecules.
- Second valley is due to electronic states of outer and then core electrons.

Jackson 315

Wavelength versus energy

$$\lambda = \frac{1240nm}{hv}$$

$$13.5nm \rightarrow 92eV$$

$$193nm \rightarrow 6.4eV$$

$$590nm \rightarrow 2.1eV$$

$$1240nm \rightarrow 1eV$$

Plasmons on metals come into this picture when $\varepsilon(\omega)$ is negative near the real axis (free electrons responding to E).

Dispersion Single Wave Propagation

$$-\bar{k}\cdot\bar{k}+\omega^2\mu(\omega)\varepsilon(\omega)=0$$

$$|\mathrm{k}|=k_0=\omega\sqrt{\mu(\omega)\varepsilon(\omega)}$$

$$k_0=\sqrt{\varepsilon_r(\omega)}\omega\mu_0\varepsilon_0$$

$$k_0=n(\omega)\frac{\omega}{c}$$

$$(k_0x-\omega t)=0$$

$$v_p=\frac{x}{t}=\frac{\omega}{k}=\frac{\omega}{n(\omega)\frac{\omega}{c}}=\frac{c}{n(\omega)}$$

- Wave equation constraint on propagation vector
- (kx-ωt)=0 => v_P=w/k
- Length of k-vector determines phase velocity
- Not all signal components remain in phase

Group Velocity Derivation

- Useful information or energy has more than one frequency to have a finite duration
- Consider Fourier Transform representation and use $\omega(k) = \omega(-k)$
- Choose a finite envelope and find frequency spread $\Delta x \Delta k > 1/2$
- Assume $\omega(k)$ approximated as linear
  - Constant => phase shift
  - $\omega d\omega/dk$ gives delay => $v_g = d\omega/dk$
- Curvature gives pulse broadening

$$v_g = \frac{c}{n(\omega) + \omega(dn/d\omega)}$$

Causality

$$\overline{D}(\overline{x},t)=\varepsilon_0\left\{\overline{E}(\overline{x},t)+\int_{-\infty}^{+\infty}G(\tau)\overline{E}(\overline{x},t-\tau)d\tau\right\}$$

$$G(\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\varepsilon(\omega)/\varepsilon_0-1\right]e^{-i\omega t}d\omega$$

$$\varepsilon(\omega)/\varepsilon_0=1+\int_0^{\infty}G(\tau)e^{i\omega\tau}d\tau$$

- D is the response to E and D(ω)=e(ω)E(ω)
- Write as summation over time
- Kernal is the polarizability and has duration of γ^{-1}.
- Must be careful when time is large and mean free path involves multiple neighbors (anomalous skin effects)
- $\varepsilon(\omega)/\varepsilon_0$ is an analytic function in the upper half-plane

Kramers-Kronig Relations

$$\mathrm{Re}\,\varepsilon(\omega)/\varepsilon_0 = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega\,\mathrm{Im}\,\varepsilon(\omega')/\varepsilon_0}{\omega'^2 - \omega^2} d\omega'$$

$$\mathrm{Im}\,\varepsilon(\omega)/\varepsilon_0 = -\frac{2\omega}{\pi} P \int_0^\infty \frac{[\mathrm{Re}\,\varepsilon(\omega')/\varepsilon_0 - 1]}{\omega'^2 - \omega^2} d\omega'$$

- Analytic => Cauchy contour intergral
- Part at infinity gives zero
- Integrate out singularity at pole
- P is principle value at pole

Boundary Conditions Create Dispersion

$$-\bar{k}\cdot\bar{k}+\omega^2\mu\varepsilon=0$$

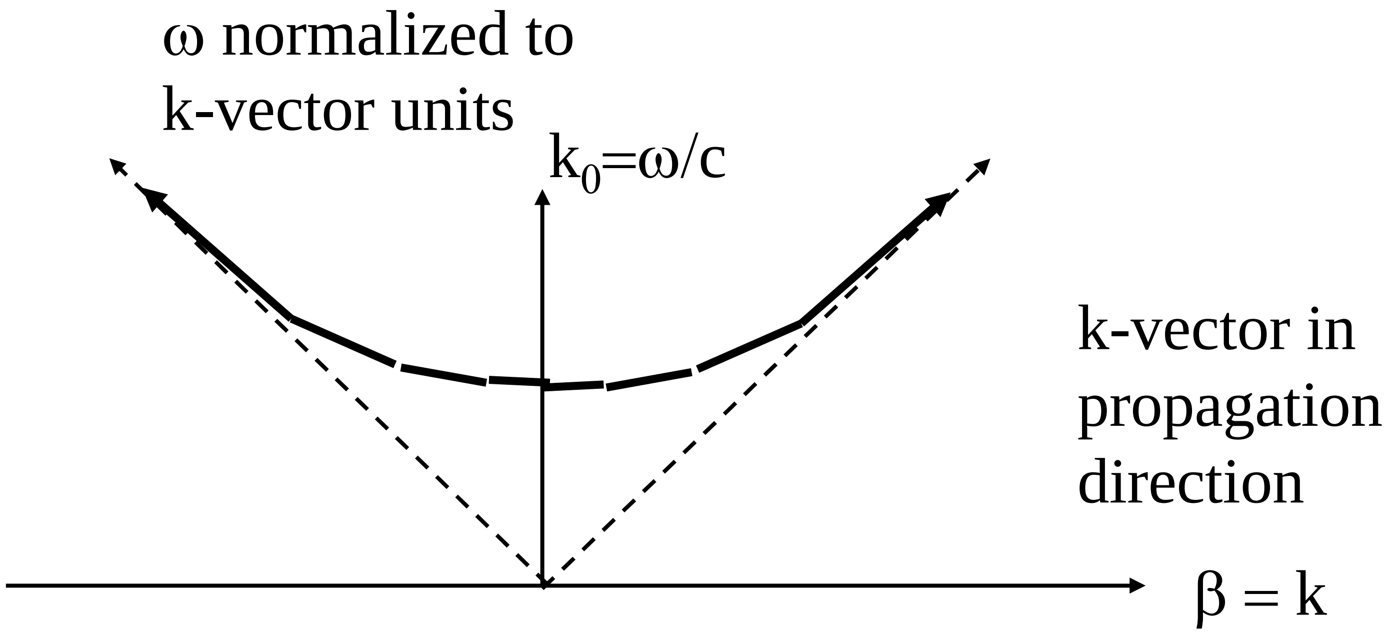
$$-k_z k_z-\frac{(2\pi)^2}{a^2}-\frac{(2\pi)^2}{b^2}+\omega^2\mu\varepsilon=0$$

$$-k_z k_z+\omega^2\mu\varepsilon[1-\frac{\frac{(2\pi)^2}{a^2}+\frac{(2\pi)^2}{b^2}}{\omega^2\mu\varepsilon}]=0$$

$$\varepsilon_r(\omega)=[1-\frac{\frac{(2\pi)^2}{a^2}+\frac{(2\pi)^2}{b^2}}{(\omega/c)^2}]$$

- Boundary conditions contribute eigenvalues
- Wave equation forces constraint that gives the dispersion relationship
- Could interpret as relative permittivity

ω-β or k-β Diagram (1-D)



- plot ω versus k dispersion relationship
- Speed of light reference $k_0 = \omega/c$
- Phase velocity = global slope
- Group velocity = local slope

Midterm Exam

- In Class Tuesday October 24th
- Covers material through today (Chapter 7)
- Open Book, Open Notes, Bring Calculator, Paper Provided
- Topics
  - Green's functions free space and use in Theorems and concepts with emphasis on statics
  - Separation of variables in rectangular coordinates using N-1 and N method
  - Time-Harmonic ME, planewaves, boundary conditions, and dispersion