EE243 Advanced Electromagnetic Theory

Lec # 11: Plane Electromagnetic Waves

- **Plane Waves in a Nonconducting Medium**
- **Linear and Circular Polarization**
- **k-space view of waves in media**
- **Reflection and Refraction at Plane Interfaces**
- **Physical Phenomena Associated with Reflection**

Reading: Jackson Ch 7.1-7.5 (skip 7.6 and 7.7)

Overview

- In a source free region for time-harmonic ($e^{-j\omega t}$) signals Maxwell's Equations can be reduced to two coupled curl equations.
- These two curl equations
  - Combine to produce the wave equation for E or H
    - The eigenfunctions for these wave equations are plane waves described by propagation direction vectors called k-vectors that have length $2\pi/\lambda$ and result in wave velocity c.
  - Have zero divergence and make the vectors E and H perpendicular to the direction of propagation.
  - Make E and H vectors perpendicular to each other
  - Are sufficient at material boundaries to
    - require the components of the k-vector parallel to the surface to be the same on both sides of the boundary (Kinematic B.C.)
    - Require tangential E and H continuous; normal D and B continuous at the boundary (Dynamic B.C.)

Time-Harmonic Maxwell Equations

Time-Varying

$$\nabla \cdot \overline{D} = \rho$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}$$

$$\nabla \cdot \overline{B} = 0$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$$

Assume
No Sources

$$e^{-i\omega t}$$

$$\overline{J} = 0$$

$$\rho = 0$$

$$\overline{D} = \varepsilon \overline{E}$$

$$\overline{B} = \mu \overline{H}$$

Time-Harmonic
Source Free

$$\nabla \times \overline{E} - i\omega \overline{B} = 0$$

$$\nabla \times \overline{B} + i\omega\mu\varepsilon \overline{E} = 0$$

Im *plicit*

$$\nabla \cdot \overline{D} = \rho$$

$$\nabla \cdot \overline{B} = 0$$

Wave Equation: Derivation

$$\nabla \times \overline{E} - i\omega \overline{B} = 0$$

$$\nabla \times \nabla \times \overline{E} - i\omega \nabla \times \overline{B} = 0$$

$$\nabla\left(\nabla \cdot \overline{E}\right) - \nabla^2 \overline{E} - i\omega\left(-i\omega\mu\varepsilon\overline{E}\right) = 0$$

$$0 - \nabla^2 \overline{E} - \omega^2 \mu\varepsilon\overline{E} = 0$$

$$\nabla^2 \overline{E} + \omega^2 \mu\varepsilon\overline{E} = 0$$

$$\nabla^2 \overline{B} + \omega^2 \mu\varepsilon\overline{B} = 0$$

- Take curl of curl E Eq.
- Sub: for curl curl
- Sub for curl B
- Use Div E = 0
- Similar Eq for B

Wave Equation: Plane Wave Solution

$$\nabla^2 \overline{E} + \omega^2 \mu\varepsilon \overline{E} = 0$$

$$\overline{E} = \overline{E}_0 e^{i\overline{k}\cdot\overline{x}}$$

$$\nabla \to i\overline{k}$$

$$\nabla \cdot \to i\overline{k}\,\cdot$$

$$\nabla \times \to i\overline{k}\,\times$$

$$\nabla^2 \to -\overline{k}^{\,2}$$

$$-\overline{k}^{\,2} + \omega^2 \mu\varepsilon \overline{E} = 0$$

$$\left|\overline{k}\right| = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

- Use 3D Fourier Expansion type eigenfunction where the vector k is the propagation vector called the k-vector
- Differential operators become algebraic operators
- Wave equation gives a constraint on the length of the k-vector
- The k-vector is reciprocal to the space variation wavelength

Plane Wave: Vector Properties

$$\overline{E} = \overline{E}_0 e^{i\overline{k}\cdot\overline{x}}$$

$$\left|\overline{k}\right| = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

$$\nabla\cdot\overline{E} = 0 \rightarrow i\overline{k}\cdot\overline{E} = 0$$

$$\nabla\cdot\overline{B} = 0 \rightarrow i\overline{k}\cdot\overline{B} = 0$$

$$\overline{B} = \frac{1}{i\omega}\nabla\times\overline{E} \rightarrow \overline{B} = \frac{1}{i\omega} i\overline{k}\times\overline{E}$$

$$\rightarrow \overline{B} = \sqrt{\mu\varepsilon}\,\hat{k}\times\overline{E}$$

$$\rightarrow \overline{H} = \frac{1}{\sqrt{\mu/\varepsilon}}\hat{k}\times\overline{E} = \frac{1}{Z_0}\hat{k}\times\overline{E}$$

$$Z_0 = \sqrt{\mu/\varepsilon} = 377 Ohms$$

- Start with a vector in 3D and variation 3D
- Div E = 0 => k perpendicular to E
- Div B = 0 => perpendicular to E
- Because E is perpendicular to k the fact that B ~ k cross E then implies B is perpendicular to E
- That is all 3 (k, B, E) are perpendicular to each other and that there are no fields in the direction of propagation

Inhomogeneous Plane Waves $|k|^2 > \omega^2\mu\varepsilon$

$$\overline{E} = \overline{E}_0 e^{i\overline{k}\cdot\overline{x}}$$

$$\overline{k} = \overline{k}_r + i\overline{k}_i$$

$$k_0 = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

$$\overline{k}\cdot\overline{k} = k_0^2$$

$$\mathrm{Re}\left(\overline{k}\cdot\overline{k}\right) = \overline{k}_r^2 - \overline{k}_i^2 = k_0^2$$

$$\mathrm{Im}\, y(\overline{k}\cdot\overline{k}) = 2\overline{k}_r\cdot\overline{k}_i = 0$$

Evanescent waves that stay near or surface and explain phenomena such as tunneling across gaps.

- The k-vector can be a vector with complex components and the imaginary part can describe exponential attenuation
- The wave equation requires the dot product with itself to
  - have a real part $\omega^2\mu\varepsilon$
  - have the imy part perpendicular to the real
- Thus the direction of maximum attenuation must be perpendicular to the direction of propagation

Plane-Wave: Poynting's Theorem

$$S = \frac{1}{2}\left(\overline{E} \times \overline{H}^*\right) = \frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}\left|E_0\right|^2$$

$$w_e = \frac{1}{4}\left(\overline{E} \cdot \overline{D}^*\right) = \frac{\varepsilon}{4}\left(\overline{E} \cdot E^*\right) = \frac{\varepsilon}{4}\left|E_0\right|^2$$

$$w_m = \frac{1}{4}\left(\overline{B} \cdot \overline{H}^*\right) = \frac{1}{4\mu}\left(\overline{B} \cdot \overline{B}^*\right) = \frac{\mu\varepsilon}{4\mu}\left|E_0\right|^2$$

$$u = \frac{1}{4}\left(\varepsilon\overline{E} \cdot E^* + \frac{1}{\mu}\left(\overline{B} \cdot \overline{B}^*\right)\right) = \frac{\varepsilon}{2}\left|E_0\right|^2$$

- Plug in Vector E with complex numbers for components
- Poynting vector, stored energy is balanced

Linear and Circular Polarization

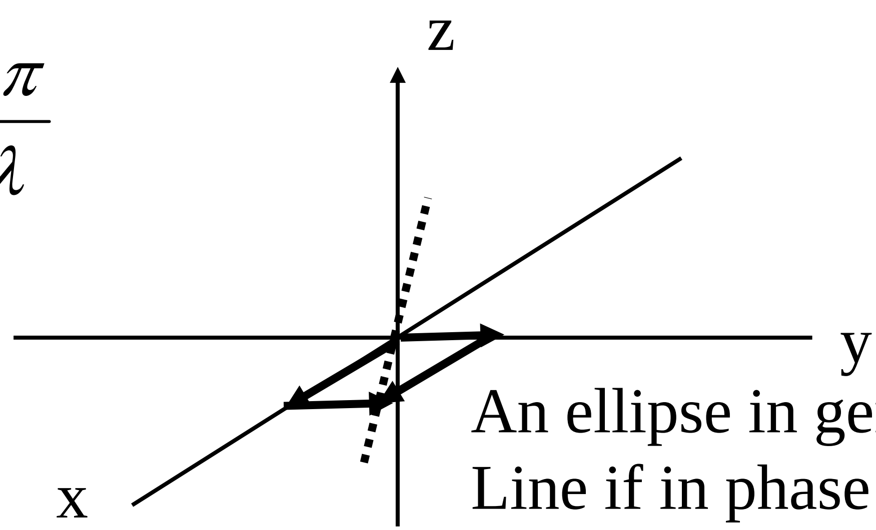
- Choose propagation in z direction and let E have components in x and y directions

$$\bar{E} = \bar{E}_0 e^{i\bar{k}\cdot\bar{x}}$$

$$\bar{E}_0 = E_{0x}\hat{x} + E_{0y}\hat{y} + E_{0z}\hat{z}$$

$$\left|\bar{k}\right| = k_0 = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

$$\bar{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$$



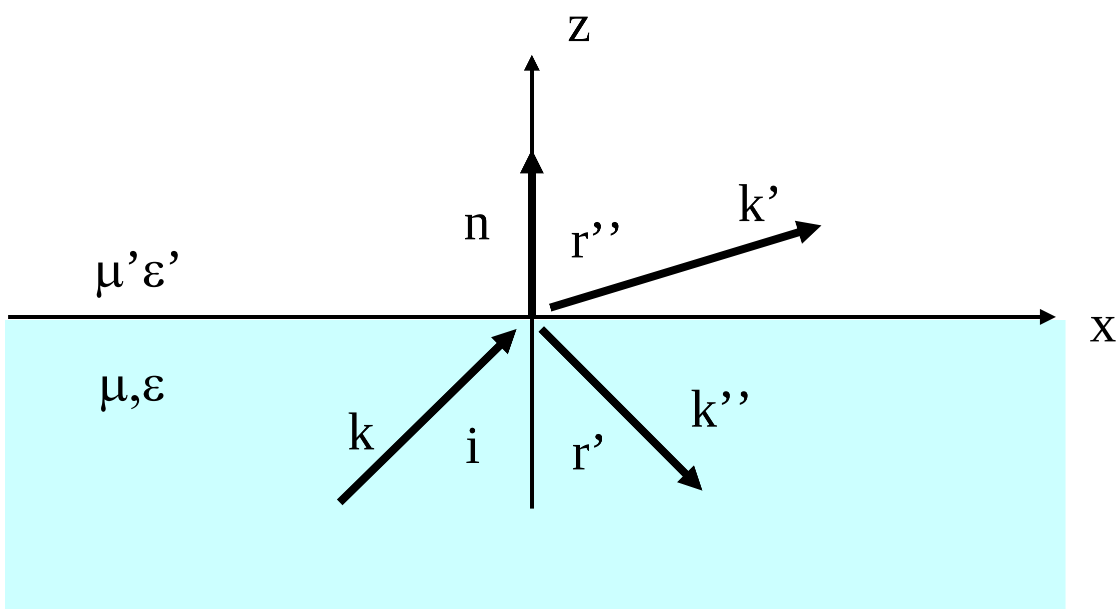
An ellipse in general
Line if in phase
Circle if out of phase
(right and left hand)

$$\bar{E}_0 = E_{0x}\hat{x} + E_{0y}\hat{y}$$

$$\bar{k} = k_0\hat{z}$$

$$\bar{E}_0 = \left|E_{0x}\right|\cos\left(k_0 z - wt + \phi_x\right)\hat{x} + \left|E_{0y}\right|\cos\left(k_0 z - wt + \phi_y\right)\hat{y}$$

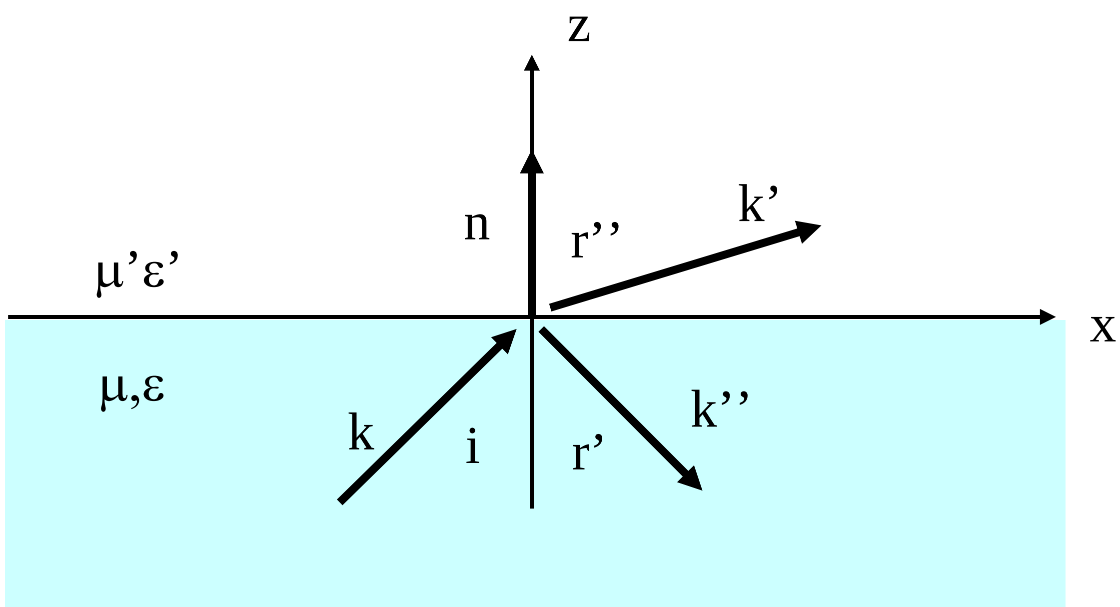
Reflection and Refraction at a Plane Interface



- Wave incident from below at angle i
- Generates transmitted (refracted) wave at angle r''
- Also generates reflected wave at angle r'

Note: The z = 0 plane where the boundary conditions are applied is for all x values and all y values

Plane Interface: Kinetic Boundary Conditions



- Since the z = 0 plane covers the range of the full Fourier representation each of the three waves must have the same eigenfunction variation along the boundary in x and y (i.e. the same k_x and k_y)

$$\left(\bar{k}\cdot\bar{x}\right)_{z=0} = \left(\bar{k}'\cdot\bar{x}\right)_{z=0} = \left(\bar{k}'\cdot\bar{x}\right)_{z=0}$$

$$k_x = k_x' = k_x'' \mapsto k_y = k_y' = k_y''$$

Plane Interface: Dynamic Boundary Conditions

- Wave Eq = 2nd order => 2 boundary conditions
- Two independent vector orientations => 4 boundary conditions toal
- Choose 4 from 6 possible and express in terms of E
  - Normal D and B continuous (2)
  - Tangential E and H continuous (4)

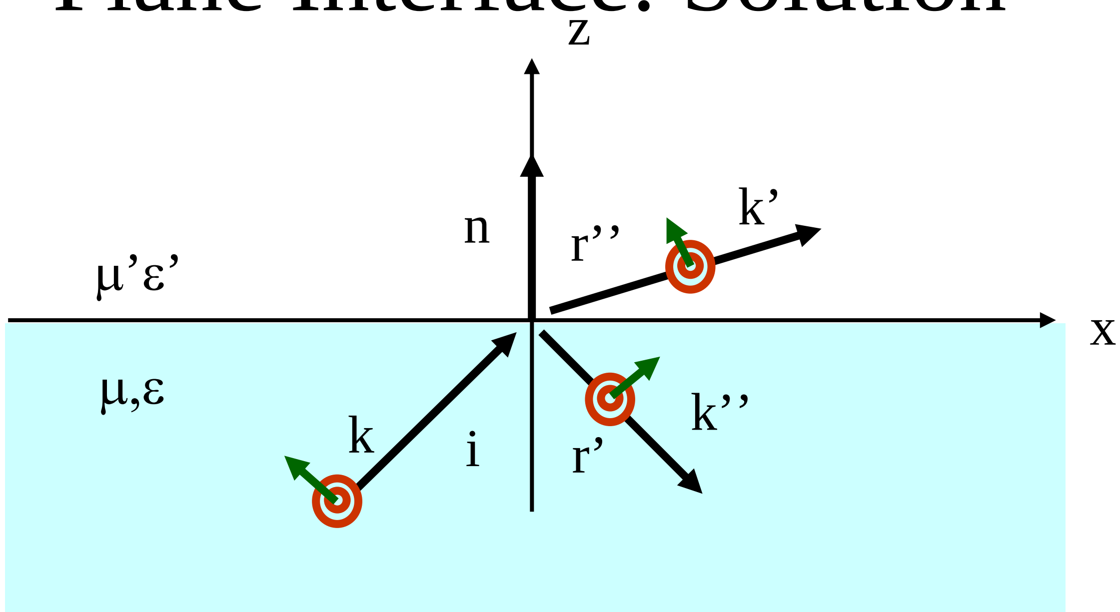
$$\left[\varepsilon\left(\left(\overline{E}_0 + \overline{E}_0''\right) - \varepsilon' \overline{E}_0'\right) - \varepsilon' \overline{E}_0'\right] \cdot \hat{n} = 0$$

$$\left[\overline{k} \times \overline{E}_0 + \overline{k}'' \times \overline{E}_0'' - \overline{k}' \times \overline{E}_0'\right] \cdot \hat{n} = 0$$

$$\left(\overline{E}_0 + \overline{E}_0'' - \overline{E}_0'\right) \times \hat{n} = 0$$

$$\left[\frac{1}{\mu}\left(\overline{k} \times \overline{E}_0 + \overline{k}'' \times \overline{E}''\right)_0 - \frac{1}{\mu'} \overline{k}' \times \overline{E}_0'\right] \times \hat{n} = 0$$

Plane Interface: Solution



- Unknowns are the transmitted and reflected wave complex amplitudes **in the plane of incidence** and **perpendicular to the plane of incident**.
- These two vector orientations can be solved independently from each other
- See Jackson Page 305 for the detailed results

Plane Interface: Physical Effects

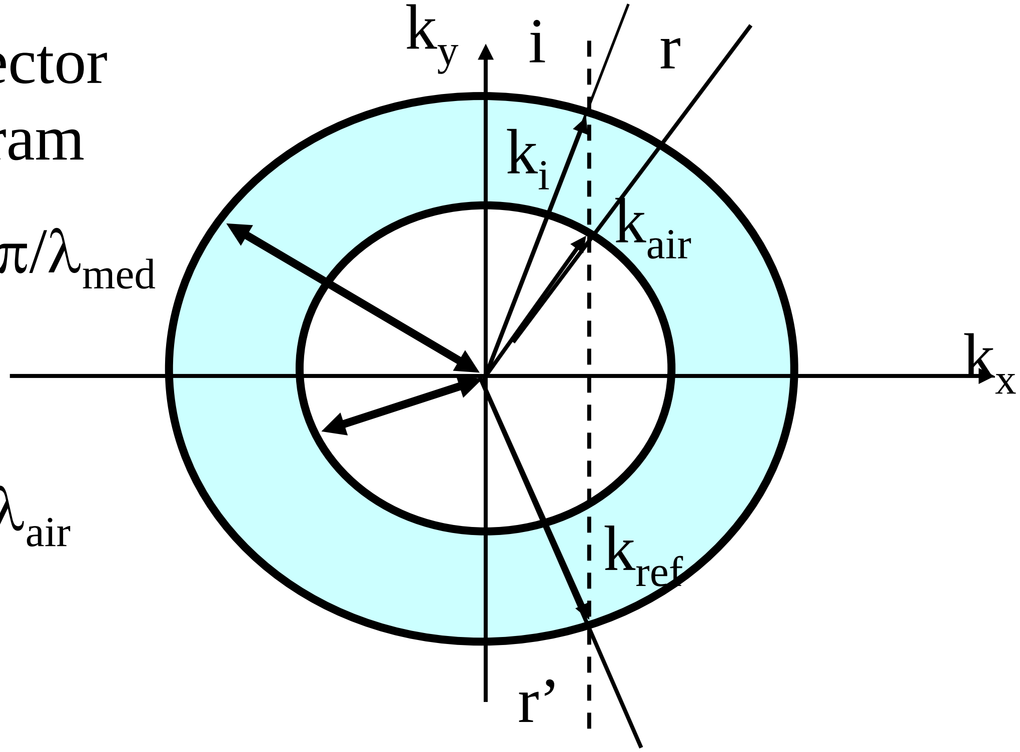
- Refraction
  - Wave direction change
- Total Internal Reflection
  - Only Evanesent fields outside
  - Tunneling accross a gap
- Brewster Angle
  - 100% transmission
- Polarization dependent phase change
  - Converts linear to part circular polarization
  - Beam spot shift (Goos-Hanchen effect)

Physical Effects: Wave Direction Change

Propagation vector
k-vector diagram

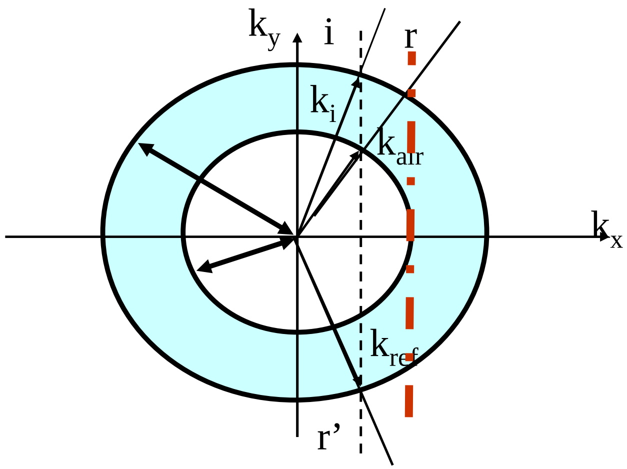
$|k_{med}| = 2\pi/\lambda_{med}$

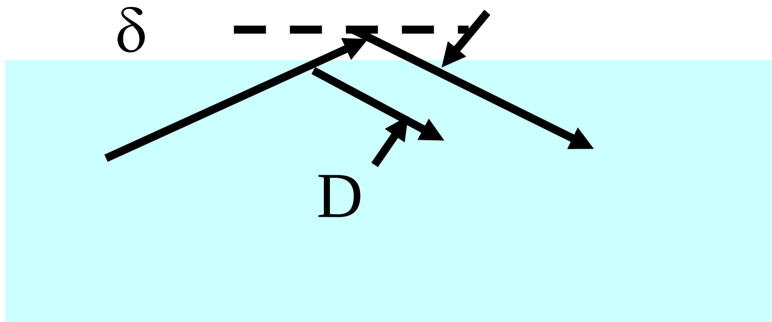
$|k_{air}| = 2\pi/\lambda_{air}$



- Draw concentric circles of radius k_{air} and k_{med}
- Incident wave has k vector given (arrow k_1)
- Find the component parallel to the surface (dotted line)
- Force the k-vector in air k_{air} and k-vector reflected k_{ref} to have the same parallel component (lie on dotted line)
- Choose point on the circle to give these new k-vectors (arrows) the correct length for the wave equation in the media that they are in

Plane Interface: Physical Effects





- Total Internal Reflection
  - Parallel part of $k_{med} > k_0$
- Brewster Angle
  - Polarization in plane of incidence reflection coefficient goes to zero giving 100% transmission
- Polarization dependent reflection phase change
  - Converts linear to part circular polarization
  - Beam energy penetration δ and spot shift D (Goos-Hanchen effect)