EE243 Advanced Electromagnetic Theory

Lec # 10: Poynting's Theorem, Time-Harmonic EM Fields

- **Poynting's Theorem**
  Conservation of energy and momentum
- **Poynting's Theorem for Linear Dispersive Media**
- **Poynting's Theorem for Time-Harmonic Fields**
- **Definition of Impedance and Admittance**
- **Foster's Reactance Theorem**
- **Lorentz Reciprocity**

Reading: Jackson Ch 6.7-6. 9 (skip 6.10)
Collin pp 2.12, 4.3, 4.4

Overview

- Starting from the work done on a current source it is possible to develop a conservation of energy that includes the flow E cross H (Poynting's Vector).
- This approach generalizes to
  - Momentum using q(E + v cross B)
  - (Phasor notation and even/odd consequences)
  - Linear dispersive media
  - Time-harmonic fields
  - Reactance has positive slope
  - Reciprocity

Work done on Source J by Field E

$$\int_V \overline{J} \cdot \overline{E} d^3 x$$

$$\int_V \overline{J} \cdot \overline{E} d^3 x = \int_V \left[\overline{E} \cdot \left(\nabla \times \overline{H} \right) - \overline{E} \cdot \frac{\partial \overline{D}}{\partial t} \right] d^3 x$$

$$\nabla \cdot (\overline{E} \times \overline{H}) = \overline{H} \cdot \left(\nabla \times \overline{E} \right) - \overline{E} \cdot \left(\nabla \times \overline{H} \right)$$

$$\int_V \overline{J} \cdot \overline{E} d^3 x = -\int_V \left[\nabla \cdot \left(\overline{E} \times \overline{H} \right) + \overline{E} \cdot \frac{\partial \overline{D}}{\partial t} + \overline{H} \cdot \frac{\partial \overline{D}}{\partial t} \right] d^3 x$$

$$\overline{J} \cdot \overline{E} = -\nabla \cdot \overline{S} - \frac{\partial u}{\partial t}$$

$$\overline{S} = \overline{E} \times \overline{H}$$

$$u = \frac{1}{2} (\overline{E} \cdot \overline{D} + \overline{B} \cdot \overline{H})$$

- Work done by fields on sources
- Replace J
- Use integration by parts like vector idenity
- Interpretation: Work done by fields on sources equals the energy flow into the voume plus the decrease in energy stored in the fields in the volume

Linear Momentum

$$\overline{F} = q(\overline{E} + v \times \overline{B})$$

$$\frac{dP_{mech}}{dt} = \int_V (\rho \overline{E} + \overline{J} \times \overline{B}) d^3x$$

$$\rho = \varepsilon_0 \overline{E}$$

$$\overline{J} = \frac{1}{\mu_0} \nabla \times \overline{B} - \varepsilon_0 \frac{\partial \overline{E}}{\partial t}$$

- Force on particle
- Momentum = force/time
- Substitute
- Many manipulations
- Integration by parts and Div theorem
- Define momentum **g**

$$\frac{dP_{mech}}{dt} + \frac{d}{dt} \int_V \varepsilon_0 (\overline{E} \times \overline{B}) d^3x = \oint_{\partial V} (terms) da$$

$$\overline{g} = \frac{1}{c^2} \overline{E} \times \overline{H}$$

$$terms = flow\_across\_boundary$$

- Interpretation: Rate of change in mechanical momentum plus rate of change in linear momentum in volume is equal to flow of mentum across the surface into the volume.

Fourier Representation Properties

$$f(x,t) = E(x,t), H(x,t), \varepsilon(x,t), etc$$

$$f(\bar{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x},\omega)e^{-i\omega t}dw$$

$$f(\bar{x},\omega) = \int_{-\infty}^{\infty} f(\bar{x},t)e^{i\omega t}dw$$

$$f(\bar{x},-\omega) = \int_{-\infty}^{\infty} f(\bar{x},t)e^{i-\omega t}dw = f^{*}(\bar{x},\omega)$$

- Here f is any function
- Fourier Representation
- Fourier Spectrum
- When f is real $f(-\omega) = f^{*}(\omega)$

Fourier Representation Implications

- Real nature of signals gives analytical properties to spectrum in the complex plane

$$\bar{E}(\bar{x},-\omega) = \bar{E}^*(\bar{x},\omega)$$

$$\bar{D}(\bar{x},-\omega) = \bar{D}^*(\bar{x},\omega)$$

$$\varepsilon(\bar{x},-\omega) = \frac{\bar{D}(\bar{x},-\omega)}{\bar{E}(\bar{x},-\omega)} = \frac{\bar{D}^*(\bar{x},\omega)}{\bar{E}^*(\bar{x},\omega)} = \varepsilon^*(\bar{x},\omega)$$

$$V(-\omega) = V^*(\omega)$$

$$I(-\omega) = I^*(\omega)$$

$$Z(-\omega) = \frac{V(-\omega)}{I(-\omega)} = \frac{V^*(\omega)}{I^*(\omega)} = Z^*(\omega)$$

Fourier Representation Implications (Cont.)

$$Z(-\omega) = \frac{V(-\omega)}{I(-\omega)} = \frac{V^*(\omega)}{I^*(\omega)} = Z^*(\omega)$$

$$Z(\omega) = R(\omega) + jX(\omega) = even + jodd$$

$$R(\omega) = \sum R_n \omega^{2n}$$

$$X(\omega) = \sum X_n \omega^{2n+1}$$

- Real nature of signals gives analytical properties to spectrum in the complex plane
- Representation for R and X contain only even and odd powers of ω
- Same is true for ε(ω)

Linear Dispersive Media

$$\overline{E}\cdot\frac{\partial\overline{D}}{\partial t}=\int d\omega\int d\omega'\left[\overline{E}(-\omega')[-i\omega\varepsilon(\omega)]\cdot\overline{E}(\omega)\right]e^{-i(\omega-\omega')t}$$

$$[\;]\rightarrow\left[\overline{E}^{*}(\omega')[-i\omega\varepsilon(\omega)]\cdot\overline{E}(\omega)\right]$$

$$[\;]\rightarrow\frac{1}{2}\left[\overline{E}^{*}(\omega')\{-i\omega\varepsilon(\omega)+i\omega'\varepsilon^{*}(\omega')\}\cdot\overline{E}(\omega)\right]$$

$$\{\;\}\rightarrow 2\left[\omega\,\mathrm{Im}(\varepsilon(\omega))-i(\omega-\omega')\frac{d}{d\omega}\left(\omega\varepsilon^{*}(\omega)\right)\right]$$

$$\overline{E}\cdot\frac{\partial\overline{D}}{\partial t}=\int d\omega\int d\omega'\left[\overline{E}(-\omega')\left[\omega\,\mathrm{Im}(\varepsilon(\omega))-i(\omega-\omega')\frac{d}{d\omega}\left(\omega\varepsilon^{*}(\omega)\right)\right]\cdot\overline{E}(\omega)\right]e^{-i(\omega-\omega')t}$$

- Constitutive relationship
- Real function constraint
- Substitute definitions using complex conjugate
- Split into two equal parts
- **Make narrowband approximation**

Linear Dispersive Media (Cont.)

$$\left\langle E \cdot \frac{\partial D}{\partial t} + H \cdot \frac{\partial B}{\partial t} \right\rangle = \omega_0 \operatorname{Im} \varepsilon(\omega_0) \left\langle \overline{E}(\overline{x},t) \cdot \overline{E}(\overline{x},t) \right\rangle$$

$$+ \omega_0 \operatorname{Im} \mu(\omega_0) \left\langle \overline{H}(\overline{x},t) \cdot \overline{H}(\overline{x},t) \right\rangle + \frac{\partial u_{eff}}{\partial t}$$

$$u_{eff} = \frac{1}{2} \operatorname{Re}\left[\frac{d(\omega\varepsilon)}{d\omega}(\omega_0) \right] \left\langle \overline{E}(\overline{x},t) \cdot \overline{E}(\overline{x},t) \right\rangle$$

$$+ \frac{1}{2} \operatorname{Re}\left[\frac{d(\omega\mu)}{d\omega}(\omega_0) \right] \left\langle \overline{H}(\overline{x},t) \cdot \overline{H}(\overline{x},t) \right\rangle$$

$$\frac{\partial u_{eff}}{\partial t} + \nabla \cdot S = -\overline{J} \cdot \overline{E} - \omega_0 \operatorname{Im} \varepsilon(\omega_0) \left\langle \overline{E}(\overline{x},t) \cdot \overline{E}(\overline{x},t) \right\rangle$$

$$- \omega_0 \operatorname{Im} \mu(\omega_0) \left\langle \overline{H}(\overline{x},t) \cdot \overline{H}(\overline{x},t) \right\rangle$$

Time-Harmonic Fields

$$e^{-i\omega t}$$

$$E(\bar{x},t) = \mathrm{Re}\left[E_0\left(\bar{x}\right)e^{i\phi\left(\bar{x}\right)}e^{-i\omega t}\right]$$

$$= E_0\left(\bar{x}\right)\cos(\phi\left(\bar{x}\right) - i\omega t)$$

$$J \cdot E = \frac{1}{2}\mathrm{Re}\left[J^* \cdot E + J \cdot Ee^{-2i\omega t}\right]$$

- E is represented by a complex number called a phasor (when it rotates)
- Products have a time independent (time-avereage) and a double frequency part

Time-Harmonic Poynting's Theorem

$$S = \frac{1}{2}\left(E \times H\right)$$

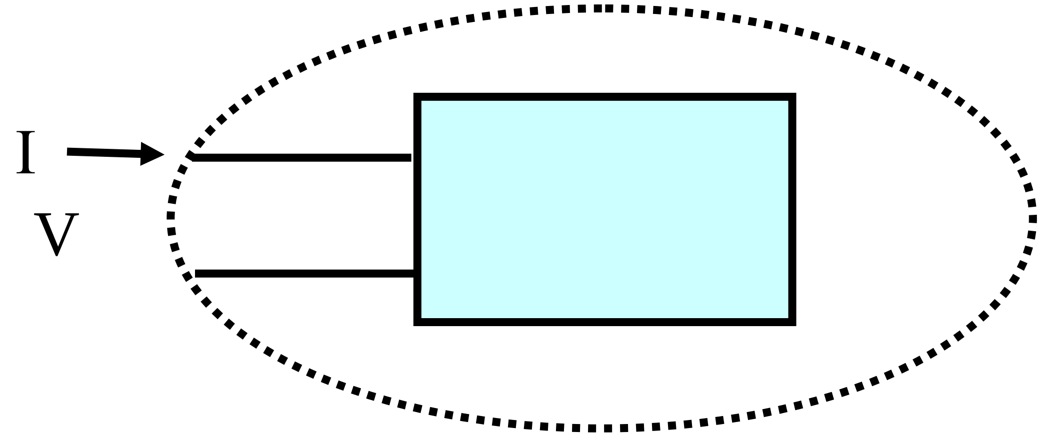
$$w_e = \frac{1}{4}\left(E \cdot D^*\right)$$

$$w_m = \frac{1}{4}\left(B \cdot H^*\right)$$

$$\frac{1}{2}\int_V J^* \cdot E d^3x + 2i\omega \int_V \left(w_e - w_m\right) d^3x + \oint_S S \cdot n da = 0$$

- Real Part = Time-average
- Imy Part is double frequency

Impedance from Poynting's Theorem



$$\frac{1}{2} I^* V = -\oint_{S_i} S \cdot n da$$

$$\frac{V}{I} = Z = R - iX = R + jX$$

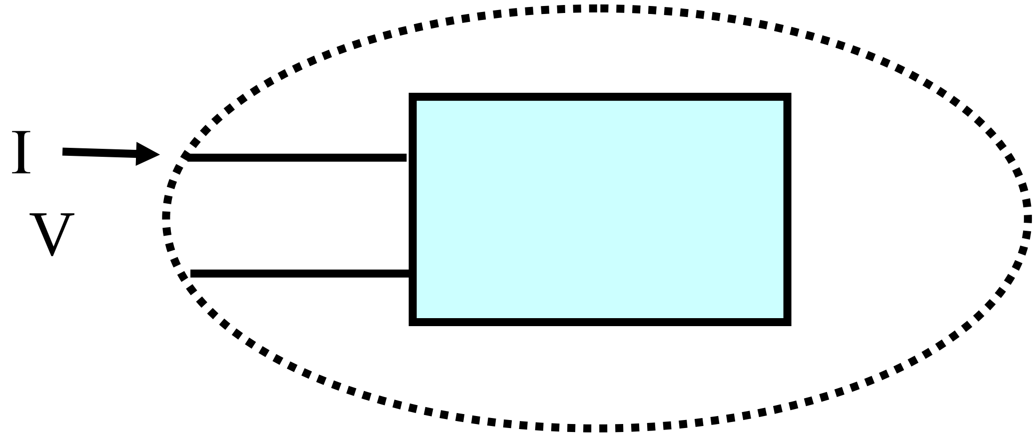
$$R - iX = \int_V J^* \cdot E d^3x + 4i\omega \int_V (w_e - w_m) d^3x + 2\oint_{S-S_i} S \cdot n da$$

- S_i is surface for signal feed and S is the outside surface
- Take Real and Imy parts for R and X

Foster's Reactance Theorem Collin 4.3

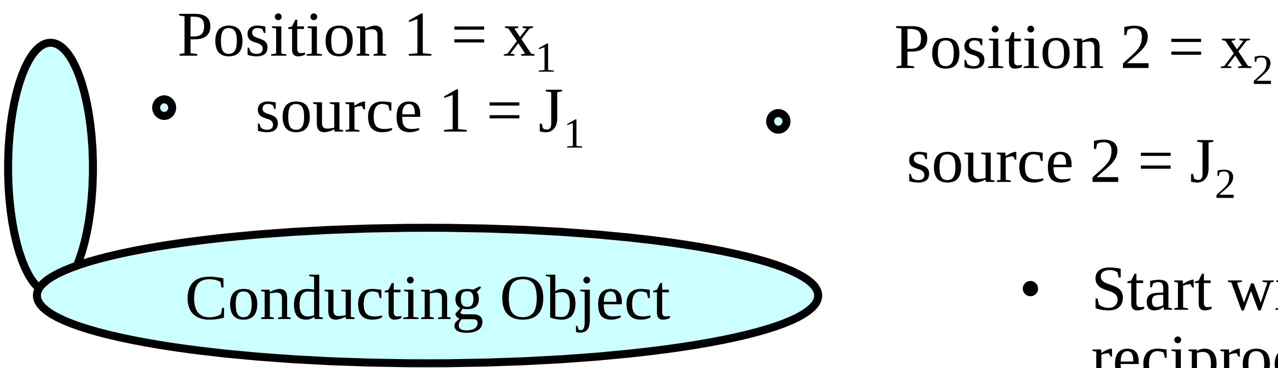
$$\frac{\partial X}{\partial \omega} = \frac{4(w_e + w_m)}{II^*}$$

$$\frac{\partial B}{\partial \omega} = \frac{4(w_e + w_m)}{VV^*}$$



- Start with div of E cross derivative of H plus derivative of E cross H; use Div theorem
- Result: The derivative of the reactance and the susceptance with respect to ω is always positive
- (There may be an alternative derivation using the time derivative of the expression for the impedance)

Lorentz Reciprocity Theorem Collin 4.3



$$-E_1(x_2)\cdot J_2(x_2)+E_2(x_2)\cdot J_2(x_2)$$

$$=\int_V\left(-J_2\cdot E_1+J_1\cdot E_2\right)dV$$

$$=\int_V\nabla\cdot\left(E_1\times H_2-E_2\times H_1\right)dV$$

$$=\oint_S\left(E_1\times H_2-E_2\times H_1\right)\cdot n\,dS$$

$$=0$$

- Start with Lorenz reciprocity statement
- put in integral form; substitute for J
- use Div theorem
- argue integral at infinity is zero due to same outgoing relationship between E and H for both sources