EE243 Advanced Electromagnetic Theory

Lec # 9: Maxwell Equations

- Maxwell Equations (including Faraday's EMF)
- Vector and Scalar Potentials
- Gauge Conditions to Give Potentials Useful Properties
- Green's Function for the Wave Equation
- Retarded Solutions
- Macroscopic Maxwell Equations Applications

Reading: Jackson Ch 6.1-6.6 (lite on 6.5 and 6.6)

Overview

- Maxwell was able to integrate Faraday's observations in to a single set of consistent equations for both statics and dynamics and Maxwell also considered light to be an electromagnetic phenomena.
- Key Ideas
  - Add displacement current
  - Re-derive the vector and scalar potential
  - Use Gauge conditions to tie down arbitrary nature
  - Use Fourier representation to find the time-retarded time-varying Green's Function solution
  - Average over molecules in 2.5 nm volume to get Macroscopic Maxwell Equations

Maxwell Equations

- Maxwell put Div D into continuity equation
- Added term is displacement current
- Boundary conditions are same as in electro- and magnetostatics

$$\nabla \cdot \overline{J} + \frac{\partial \rho}{\partial t} = \nabla \cdot \left(\overline{J} + \frac{\partial \overline{D}}{\partial t} \right) = 0$$

$$\rightarrow\rightarrow\rightarrow\rightarrow$$

$$\nabla \cdot \overline{D} = \rho$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}$$

$$\nabla \cdot \overline{B} = 0$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$$

Auxiliary Mathematical Potentials now Both Scalar and Vector

- Div B = 0 allows B to be represented by curl **A**
- Curl (E plus time derivative of A) = 0 says that this quantity can be described by Gradient Φ
- Thus both **A** and Φ are required.

$$\nabla \cdot \overline{B} = 0$$

$$\overline{B}(\overline{x}) = \nabla \times \overline{A}(\overline{x})$$

$$\nabla \times \left(\overline{E} + \frac{\partial \overline{A}}{\partial t} \right) = 0$$

$$\overline{E} = -\nabla \Phi - \frac{\partial \overline{A}}{\partial t}$$

Wave Equations for Auxiliary Mathematical Potentials **A** and Φ under Lorentz Gauge

$$\nabla^2\Phi + \frac{\partial}{\partial t}\left(\overline{A}\right) = -\rho / \varepsilon_0$$

$$\nabla^2\overline{A} - \frac{1}{c^2}\frac{\partial^2\overline{A}}{\partial t^2} - \nabla\left(\underline{\nabla\cdot\overline{A} + \frac{1}{c^2}\frac{\partial\ \Phi}{\partial t}}\right) = -\mu_0\overline{J}$$

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\rho / \varepsilon_0$$

$$\nabla^2\overline{A} - \frac{1}{c^2}\frac{\partial^2\overline{A}}{\partial t^2} = -\mu_0\overline{J}$$

- Lorentz Condition (Gauge) when term in this bracket is zero
- Potentials are still not unique

Coulomb Potential and Transverse Current for Auxiliary Mathematical Potentials **A** and Φ under **Div A = 0**

$$\nabla \cdot \overline{A} = 0$$

$$\nabla^2 \Phi = -\rho / \varepsilon_0$$

$$\nabla^2 \overline{A} - \frac{1}{c^2} \frac{\partial^2 \overline{A}}{\partial t^2} = -\mu_0 \overline{J} + \frac{1}{c^2} \nabla \frac{\partial \Phi}{\partial t}$$

$$\overline{J} = \overline{J}_l + \overline{J}_t$$

$$\nabla^2 \overline{A} - \frac{1}{c^2} \frac{\partial^2 \overline{A}}{\partial t^2} = -\mu_0 \overline{J}_t$$

- Coulomb Gauge Div A = 0
- Φ is instantaneous near field
- Longitudinal and transverse current
- Only transverse current radiates
- Balloon coated with charge and with oscillating radius does not radiate

Green's Function for the Wave Equation Based on Fourier Representation

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi f(\bar{x},t)$$

$$\Psi(\bar{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(\bar{x},\omega)e^{-i\omega t}dw$$

$$f(\bar{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x},\omega)e^{-i\omega t}dw$$

$$\Psi(\bar{x},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(\bar{x},t)e^{i\omega t}dw$$

$$f(\bar{x},w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x},t)e^{i\omega t}dw$$

- Typical Wave Equation
- Fourier Representation
- Fourier Spectrum
- Apply to both source distribution and unknown function

Green's Function for the Wave Equation Radial Behavior

$$\left(\nabla^2 + k^2\right)\Psi(\bar{x},\omega) = -4\pi f(\bar{x},\omega)$$

$$\left(\nabla^2 + k^2\right)G(\bar{x},\omega) = -4\pi\delta(\bar{x}-\bar{x}')$$

$$\frac{1}{R}\frac{d^2}{dR^2}(RG_k) + k^2 G_k = -4\pi\delta(\bar{R})$$

$$\frac{d^2}{dR^2}(RG_k) + k^2 RG_k = 0$$

$$RG_k(R) = Ae^{ikR} + Be^{-ikR}$$

$$G_k^{\pm}(R) = \frac{e^{\pm ikR}}{R}$$

- Typical Wave Equation
- Green's Function
- Boundary free case can only depend on R
- Diff Eq. for R variation
- Normalized
- Outward is -kR

Green’s Function for the Wave Equation Time-Retarded (for propagation)

$$\left(\nabla^2 + k^2\right) G(\bar{x}, \omega) = -4\pi\delta(\bar{x} - \bar{x}')\delta\left(t - t'\right)$$

$$\delta\left(t - t'\right) \rightarrow e^{-i\omega t'}$$

$$G^{\pm}(\bar{x}, t, \bar{x}', t') = \frac{\delta\left(t' - \left[t \mp \frac{\left|\bar{x} - \bar{x}'\right|}{c}\right]\right)}{\left|\bar{x} - \bar{x}'\right|}$$

- Put in time delta
- Fourier Transform
- $\tau = t - t'$
- Transform back
- Use + for source genera ration

This time retardation applies to the vector potential under both the Lorentz and Coulomb Gauges and to the scalar potential only under the Lorenz Gauge

Retarded Solutions for B and E

- Apply operators to get E and B
  - Jefimenko expressions
- Work out the retarded derivatives
  - Heaviside-Feynman expressions
  - E ~q{radial}
  - B ~q{v cross **R**}

Derivation of Macroscopic Equations

- Average over space and time
  - Volume 2.5 nm on a side 1000 atoms
  - Time longer than dielectric relaxation time 10^{-14} s
- Proceedure
  - Microscopic equations
  - Tapered support or finite support
  - Free and Bound charges
  - Molecular multipole moments
  - Equal to a collection of point multipoles

Macroscopic Maxwell Equations

- Same Equations
- Constitutive Relationships
- Propagation parameter

$$\nabla \cdot \overline{D} = \rho$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}$$

$$\nabla \cdot \overline{B} = 0$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$$

$$\overline{D} = \varepsilon \overline{E}$$

$$\overline{B} = \mu \overline{H}$$

$$k = \omega \sqrt{\mu / \varepsilon}$$