EE243 Advanced Electromagnetic Theory

Lec # 8: Magnetostatics

- **Basic Observations of Magnetic Fields and Forces**
- **Vector Potential**
- **Magnetic Moment Density (magnetization)**
- **Force, Torque, Energy**
- **Macroscopic Equations and Boundary Conditions**
- **Applications**

Reading: Jackson Ch 5

Overview

- Since there are no magnetic charges, one might think Magnetostatics would be easier.
- But no magnetic charges has the opposite effect of greatly complicating the physics and the math.
  - Start with dipole (higher order) effects
  - Now think in terms of fields that circle rather than diverge (lots of cross products)
  - The potential which for electrostatics was a scalar with a clear physical interpretation becomes a multi-component vector without an intuitive interpretation

Basic Observations

- Statics means charge does not change with time => div J = 0
  $$\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0$$
  $$\rightarrow_{STATICS} \nabla \cdot J = 0$$
- Measure torque
  $$N = \mu \times B$$
- Biot and Savart Law (1820)
  $$d\overline{B} = \frac{\mu_0}{4\pi} I \frac{\left(d\bar{l} \times \bar{x}\right)}{\left|\bar{x}\right|^3}$$
- Generalize for charge in motion
  $$\overline{B} = \frac{\mu_0}{4\pi} q \frac{\bar{v} \times \bar{x}}{\left|\bar{x}\right|^3}$$
- Force on current element
  $$d\overline{F} = I_1 \left(d\bar{I}_1 \times \overline{B}\right)$$

Basic Observations

- B field from a long wire at distance R
  $$\left|\overline{B}\right| = \frac{\mu_0}{4\pi} I \int_{-\infty}^{\infty} \frac{(dl)}{\left(R^2 + l^2\right)^{3/2}} = \frac{\mu_0}{2\pi} \frac{I}{R}$$
- Force on a closed current loop due to another closed current loop
  $$\overline{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{\left(d\bar{I}_1 \cdot d\bar{I}_2\right)\bar{x}_{12}}{\left|\bar{x}_{12}\right|^3}$$
- Force on parallel wires repel if opposite direction
  $$\frac{dF}{dl} = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d}$$
- Force on a current distribution
  $$\overline{F} = \int \bar{J}(\bar{x}) \times \overline{B}(\bar{x}) d^3x$$
- Torque on a current distribution
  $$\overline{N} = \int \bar{x} \times \left(\bar{J} \times \overline{B}\right) d^3x$$

Differential Form of Magnetostatics

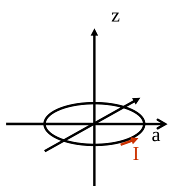
- Start from current element and integrate
  $$\overline{B}(\bar{x}) = \frac{\mu_0}{4\pi} \int \bar{J}(\bar{x}') \times \frac{\bar{x} - \bar{x}'}{\left|\bar{x} - \bar{x}'\right|^3} d^3x$$
- Get curl integral
  $$\overline{B}(\bar{x}) = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\bar{J}(\bar{x}')}{\bar{x} - \bar{x}'} d^3x$$
- Implies div B = 0
  $$\nabla \cdot \overline{B} = 0$$
- Go back to integral and take curl
  $$\nabla \times \overline{B} = \frac{\mu_0}{4\pi} \nabla \times \nabla \times \int \frac{\bar{J}(\bar{x}')}{\bar{x} - \bar{x}'} d^3x$$
- Curl B = μ_0J
  $$\nabla \times \overline{B} = \mu_0 \bar{J}$$
- Ampere's Law for loop of B field = μ_0 current
  $$\oint \overline{B} \cdot d\bar{l} = \mu_0 \int \bar{J} \cdot \hat{n} da = \mu_0 I$$

Auxiliary Mathematical Potential now Vector

- Div B = 0 allows B to be represented by curl A
  $$\nabla \cdot \overline{B} = 0$$
  $$\overline{B}(\bar{x}) = \nabla \times \overline{A}(\bar{x})$$
- Previous slide shows an A
  $$\overline{A}(\bar{x}) = \frac{\mu_0}{4\pi} \int \frac{\bar{J}(\bar{x}')}{\left|\bar{x} - \bar{x}'\right|} d^3x' + \nabla\Psi(\bar{x})$$
- But arbitrary grad ψ can be added
  $$\nabla \cdot \overline{A} = 0 \rightarrow \nabla^2 \Psi(\bar{x}) = 0$$
  $$\rightarrow \Psi(\bar{x}) = C$$
- Choose Div A =0 Coulomb Gauge
  $$\nabla^2 \overline{A} = -\mu_0 \bar{J}$$

Vector Potential Example: Circular Loop

- Current I on circular loop of radius a
- Integrate J to get A
  - => A_ϕ only
- Apply curl to get B
- Look at far fields
  - Dipole in nature
  - Magnitude m = $\pi a^2 I$ = area times I



$$\overline{A}(x) = \frac{\mu_0}{4\pi}\frac{\overline{m}\times\overline{x}}{|x|^3}$$

$$\overline{B}(x) = \frac{\mu_0}{4\pi}\left[\frac{3\hat{n}(\hat{n}\cdot\overline{m})-\overline{m}}{|x|^3}\right]$$

Force, Torque and Potential Energy

- Force
  - Example: Magnetic Mirror
- Potential
- Torque
  - Example: Motors

$$\overline{F} = \nabla(\overline{m}\cdot\overline{B})$$

$$U = -\overline{m}\cdot\overline{B}$$

$$\overline{N} = -\overline{m}\cdot\overline{B}(0)$$

Material Model

- Circular currents due to charge particles orbiting nucleus
  - Sum over particles
  - Related to angular momentum
  - Also quantum effects
  - Local magnetization

$$\overline{m} = \frac{1}{2}\sum_i q_i(\overline{x}_i \times \overline{v}_i)$$

$$\overline{L}_i = M_i(\overline{x}_i \times \overline{v}_i)$$

$$\overline{m} = \frac{e}{2M}\overline{L}$$

$$\overline{M}(\overline{x}) = \sum_i N_i\langle \overline{m}_i\rangle$$

Microscopic to Macroscopic

- Average of div B = 0 scales
- Vector potential A has magnetization contributions (dipole like)
- Magnetization contributes an effective current density
- Introduce new macroscopic magnetic field
- Curl H = J

$$\nabla\cdot\overline{B} = 0$$

$$\overline{A}(\overline{x}) = \frac{\mu_0}{4\pi}\int\frac{\overline{J}(\overline{x}')+\nabla'\times\overline{M}(\overline{x}')}{|\overline{x}-\overline{x}'|}d^3x'$$

$$\overline{J} = \nabla\times\overline{M}$$

$$\nabla\times\overline{B} = \mu_0[\overline{J}+\nabla\times\overline{M}]$$

$$\overline{H} = \frac{1}{\mu_0} - \overline{M}$$

$$\nabla\times\overline{H} = \overline{J}$$

Boundary Conditions

$$(\overline{B}_2 - \overline{B}_1)\cdot\hat{n}_{21} = 0$$

$$\hat{n}_{21}\times(\overline{H}_2 - \overline{H}_1) = \overline{K}_{SURFACE}$$

$$\hat{n}_{21} = from\_1\_to\_2$$

- Derived from div B = 0 and curl H = J
- Very large values of permeability occur in which case
  - H is nearly normal to the surface where the B field emerges from the material
  - H is very small inside the material
    - Example: electromagnet with N turn coil; Integral of H in loop = NI; and H times gap ~ NI

Boundary Value Problem Techniques

- Choice of convenient Auxiliary function to represent B
- Vector Potential A and B = Curl A
  - Integrate the vector current with 1/distance to get vector A; take curl to get B; divide μ to get B; apply boundary conditions
- Magnetic Scalar Potential Φ and H =- Grad Φ
  - Special case of no local currents to integrate
  - Curl H =0 => H =- Grad Φ
  - Choose potential; Grad to get H, multiply μ to get B; apply boundary conditions

Example BVP in Magnetostatics

- Uniformly magnetized sphere
  - m = volume times M
- Magnetized sphere in and external field
  - Analogous to electrostatics
  - Internal Magnetization
- Magnetic Shielding
  - High μ material attracts nearly all field lines
- Hole in a conducting plane in uniform magnetic field
  - Dipole in direction of field

Faraday's Law of Induction

- Flux through surface $F = \int_S \overline{B} \cdot \hat{n} da$
- Electro Motive Force that opposes is proportion to rate of change $Emf = \oint_C \overline{E}' \cdot d\overline{l}$ $\quad Emf = -\frac{dF}{dt}$
- Include velocity of moving loop $\oint_C \left[\overline{E}' - (\overline{v} \times B) \right] \cdot d\overline{l} = -\oint_S \frac{\partial \overline{B}}{\partial t} \cdot \hat{n} da$
- Differential form $\nabla \times \overline{E} + \frac{\partial \overline{B}}{\partial t} = 0$

Energy in Magnetic Fields

- Power = VI
- Small loop
- A and J in space
- B and H in space (most physical)
- Force is gradient of W at constant current

$$\frac{dW}{dt} = -IEmf = I \frac{dF}{Dt}$$
$$\delta W = I \delta F$$
$$\delta W = J \Delta \sigma \int \hat{n} \cdot \delta \overline{B} d^3x$$
$$\delta W = \int \delta \overline{A} \cdot \overline{J} d^3x$$
$$\delta W = \int \delta \overline{B} \cdot \overline{H} d^3x$$

$$W = \frac{1}{2} I \Delta F$$
$$W = \frac{1}{2} \int \overline{H} \cdot \overline{B} d^3x$$
$$W = \frac{1}{2} \int \overline{J} \cdot \overline{A} d^3x$$
$$W = \frac{1}{2} \int \overline{M} \cdot \overline{B} d^3x$$
$$\overline{F} = (\nabla W)_J$$

Self- and Mutual Inductances

- Define using analogy to capacitance
- Sort formula for energy into current loops
- When loop small compared to change in vector potential over it the flux linkage can be used

$$W = \frac{1}{2} \sum_{i=1} L_i I^2{}_i + \sum_{i=1}^{N} \sum_{j>i}^{N} M_{ij} I_i I_j$$
$$W = \frac{\mu_0}{8\pi} \int d^3x \int d^3x' \frac{\overline{J}(\overline{x}) \cdot \overline{J}(\overline{x}')}{|\overline{x} - \overline{x}'|}$$
$$L_{ij} = \frac{\mu_0}{4\pi I^2{}_i} \int_{C_i} d^3x_i \int_{C_i} d^3x_i \frac{\overline{J}(\overline{x}_i) \cdot \overline{J}(\overline{x}_i')}{|\overline{x} - \overline{x}'|}$$
$$M_{ij} = \frac{\mu_0}{4\pi I_i I_j} \int_{C_i} d^3x_i \int_{C_j} d^3x_j \frac{\overline{J}(\overline{x}_j) \cdot \overline{J}(\overline{x}_j')}{|\overline{x}_i - \overline{x}_j'|}$$
$$M_{ij} = \frac{\mu_0}{I_i} F_{ij}$$