EE243 Advanced Electromagnetic Theory

Lec # 6: Boundary Value Problems (Cont.)

- **Finish Separation of Variables in N-1 Dimensions**
- **Separation of Variables N Dimensions (Eigenfunction)**
- **{Help with Homework Sets 3 and 4}**

Reading: Jackson
2.11 and 3.12

Separation of Variables: Representation

Jackson 2.9

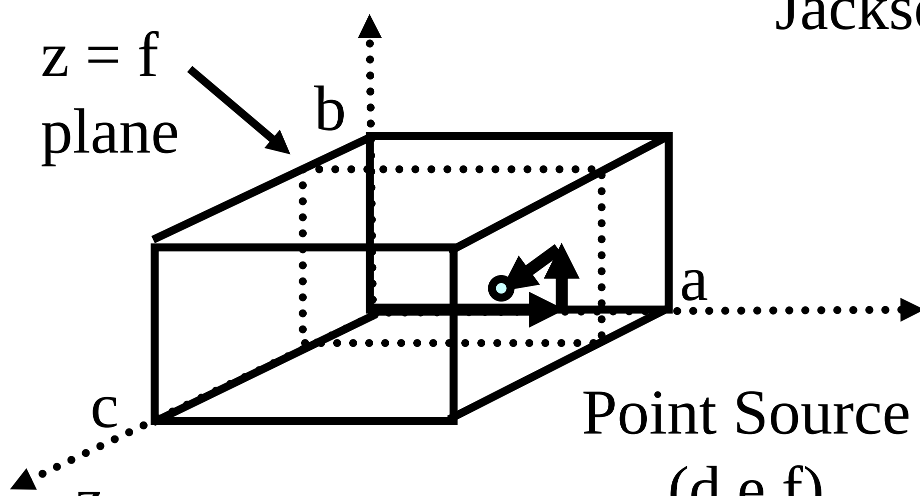
$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z)$$

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} B_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} (c - z))$$

- Sum is over composite eigenvalues in N-1 dimensions
- Note that the boundary conditions for x=0, x = a and for y=0 and y= b are met by sin behavior
- Note that the boundary conditions at z = 0 and z = c have already been applied in sinh behavior.

Separation of Variables: Source Strategy

Jackson 2.9



- View source as being on z = f plane.
- Require Φ_2 - Φ_1 =D(x,y)/ε_0 at z = f
- Also require at z =f

$$(\bar{E}_2 - \bar{E}_1)\cdot\hat{n} = \sigma_{SURFACE}(x,y)/\varepsilon_0$$

- Multiply each of these equations by one of the composite eigenfunctions and integrate over x,y cross-section
- Gives two equations relating A_{nm} and B_{nm} for the same nm.

$$\nabla^2\psi = -\frac{4\pi q}{\varepsilon_0}\delta(x-d)\delta(y-e)\delta(z-f)$$

Separation of Variables: Source Results

$$A_{nm}\sinh(\gamma_{nm}f)+B_{nm}\sinh(\gamma_{nm}(c-f))=0$$

$$A_{nm}\cosh(\gamma_{nm}f)-B_{nm}\cosh(\gamma_{nm}(c-f))=\sigma_{nm}$$

$$\sigma_{nm}=\frac{4}{ab}\int_0^a\int_0^b\sigma(x,y)\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)dxdy$$

$$\sigma(x,y)=-\frac{4\pi q}{\varepsilon_0}\delta(x-d)\delta(y-e)$$

Jackson 3.12

$$\sigma_{nm}=\frac{q16\pi}{\varepsilon_0 ab}\sin\left(\frac{2\pi d}{a}\right)\sin\left(\frac{2\pi e}{a}\right)$$

- The delta function source makes the source integral and expansion trivial

Separation of Variables: Final Result

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)$$

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} B_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm}(c-z))$$

Jackson 3.12

- Solve for A_{nm} and B_{nm} and plug in
- Both proportional to σ_{nm}
- Also involve ratios of sinh and cosh
- See 3.168 pp 129

Separation of Variables: Final Result

Jackson 3.12

$$\Phi(x,y,z) = \frac{q16\pi}{ab}\sum_{n,m=1}^{\infty} A_{nm}\sin(\alpha_n x)\sin(\alpha_n d)\sin(\beta_m y)\sin(\beta_m e)$$

$$\times \frac{\sinh(\gamma_{nm} z_{\langle})\sinh(\gamma_{nm}(c - z_{\rangle}))}{\gamma_{nm}\sinh(\gamma_{nm} c)}$$

$$\gamma_{nm} = \pi(l^2/a^2 + m^2/b^2)^{1/2}$$

$z_<$ is the smaller of z and f
$z_>$ is the larger of z and f

Separation of Variables: Eigenfunctions

Jackson 3.12

$$\nabla^2 \psi(\bar{x}) + [f(\bar{x}) + \lambda]\psi(\bar{x}) = 0$$

$$\nabla^2 \psi_n(\bar{x}) + [f(\bar{x}) + \lambda_n]\psi_n(\bar{x}) = 0$$

$$\int_V \psi_m^*(\bar{x})\psi_n(\bar{x})d^3x = \delta_{nm}$$

- Differential Equation in 3D
- Find eigenfunctions and eigenvalues
- Eigenfunctions are orthogonal and then normalized

Separation of Variables: Greens Function

$$\nabla_x^{\,2} G(\bar{x},\bar{x}') + [\,f(\bar{x}) + \lambda]G(\bar{x},\bar{x}') = -4\pi\delta(\bar{x}-\bar{x}')$$

$$G(\bar{x},\bar{x}') = \sum_n a_n(\bar{x}')\psi_n(\bar{x})$$

$$\sum_m a_m(\bar{x}')(\lambda - \lambda_n)\psi_m(\bar{x}) = -4\pi\delta(\bar{x}-\bar{x}')$$

$$a_m(\bar{x}') = \frac{4\pi\psi_n(\bar{x}')}{\lambda_n - \lambda}$$

$$G(\bar{x},\bar{x}') = \sum_n \frac{4\pi\psi_n(\bar{x}')\psi_n(\bar{x})}{\lambda_n - \lambda}$$

- Substitute Eigenfunction expansion for G into PDE and solve

Separation of Variables: ND Example

$$(\nabla^2 + k_{lmn}^2)\psi_{lmn}(x,y,z) = 0$$

$$\psi_{lmn}(x,y,z) = \sqrt{\frac{8}{abc}}\sin(\frac{l\pi x}{a})\sin(\frac{m\pi y}{b})\sin(\frac{n\pi z}{c})$$

$$k_{lmn}^2 = \pi^2(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2})$$

$$G(\overline{x},\overline{x}') = \frac{32}{\pi abc}\sum_{l,m,n=1}^{\infty}\frac{\sin(\frac{l\pi x}{a})\sin(\frac{l\pi x'}{a})\sin(\frac{m\pi y}{b})\sin(\frac{m\pi y'}{b})\sin(\frac{n\pi z}{c})\sin(\frac{n\pi z'}{c})}{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}}$$

- Expansion is in 3 dimensions
- Result for a point source is very symmetric in x and x’

Separation of Variables: Generalization

- What if box has voltages on walls?
- What if Dipole layer?
- What if the source is distributed on a plane?
- What if source is distributed in space?
- What if electrodynamics instead of statics?