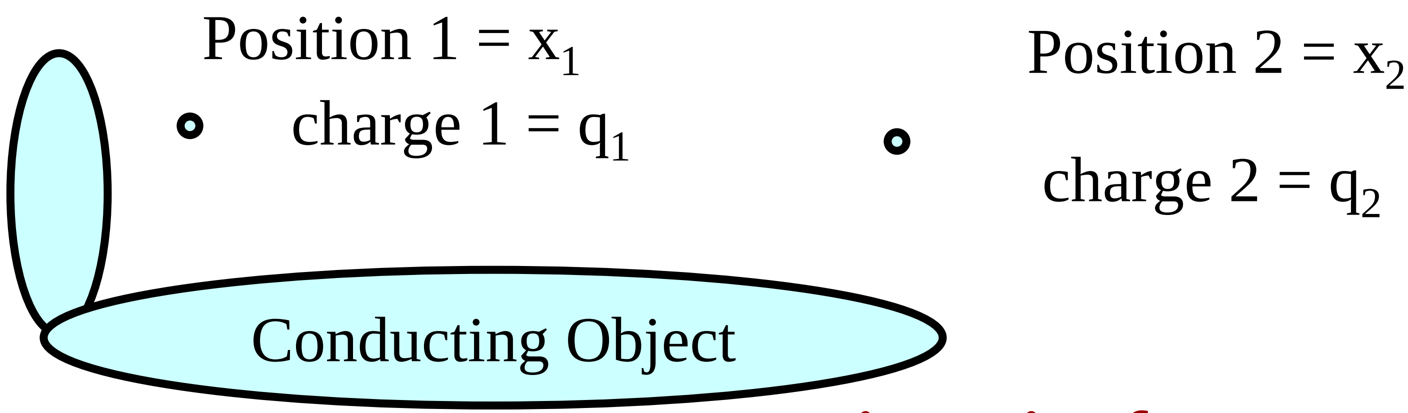
EE243 Advanced Electromagnetic Theory

Lec # 5: Boundary Value Problems

- Touch up Reciprocity, Variation, Finite-Element
- Orthogonal functions
- Constant Product of Widths (Space x Spectrum)
- Initial-Final Asymptotic Behavior
- Summation of Complex Series
- Start Separation of Variables N-1 Dimensions

Reading: Jackson
2.8-2.12, 3.4

Reciprocity



$\Phi(x_2,q_1) = \Phi(x_1,q_2)$

Proof:

- Green’s Theorem
- Poisson’s equation for $\Phi(x_2,q_1)$ and $\Phi(x_1,q_2)$ causes volume integral to give $\Phi(x_2,q_1)$ - $\Phi(x_1,q_2)$
- In surface integral use homogeneous boundary condition to replace potential with derivative and integrand vanishes at every point on the boundary

Reciprocity for Green’s Function in Jackson

$$G(\bar{x}, \bar{x}') = (\bar{x}', \bar{x})$$

Variational Approaches

Jackson 1.12

$$I[\psi] = \frac{1}{2}\int_V \nabla\psi \cdot \nabla\psi d^3x - \int_V g\psi d^3x - \oint_S f\psi da$$

- Energy like functionals are useful as physical systems have minimal energy corresponding to minimizing these functionals
- The above functional has
  - energy stored in fields in volume
  - Minus work done on sources g in volume
  - Minus energy flow away across the boundary
- Look at change $\psi \to \psi + \delta\psi$
  - Require δI vanish independent of change
  - Gives Poisson's equation source g and $\frac{\partial\psi}{\partial n} = f$

Finite Element Methods

Jackson 2.12 2D Example

$$\int_R [\phi\nabla^2\psi + g\phi]dxdy = 0 \qquad \int_R [\nabla\phi\cdot\nabla\psi - g\phi]dxdy = 0$$

- Here $\phi(x,y)$ is a test function that is zero on boundary (Dirichlet)
- This boundary condition makes the integrals equal
- Choose $\phi_{ij}(x,y)$ linear on rectangle or triangle i,j and zero elsewhere and express in 4 or 3 node values
- Represent solution: Cover domain $\psi(x,y) \approx \sum_{k,l}^{N_0} \psi_{k,l}\phi_{k,l}(x,y)$
- Put this representation into the right hand integral and let $\phi = \phi_{ij}(x,y)$
- Repeat for each rectangle or triangle and get one equation that is sparse in node values
- Solve for node values

Orthonormal Functions and Expansions

Jackson 2.8

- Ortohnormal functions
- Approx. Sum
- Mean Square Min.
- Coefficient
- Convergs to the mean at discontinuities
- Completeness
- Mean Square

$$\int_a^b U_n^*(\xi) U_m(\xi) d\xi = \delta_{nm}$$

$$f(x) \leftrightarrow \sum_{n=1}^{N} a_n U_n(\xi)$$

$$M_N = \int_a^b \left| f(\xi) - \sum_{n=1}^{N} a_n U_n(\xi) \right|^2 d\xi$$

$$a_n = \int_a^b U_n^*(\xi) f(\xi) d\xi$$

$$MS = \sum_{n=0} a_n^2$$

Fourier Series Example

Jackson 2.8 pp 68

- Interval –a/2 to a/2
- Normalized sqrt(2)/a sin(2πmx/a) and cos(2πmx/a) plus constant
- f(x) = 2/a integral f(x) times sin or cos

Fourier Integral Example

Jackson 1.12 pp. 69

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(k) e^{ikx} dk$$

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

- Infinite domain => continuous distribution
- A(k) = spectral distribution or spectrum

Constant Space Bandwidth Product

Horse Sense to check Answers

Jackson pp. 324

- Let Δx = rms deviation f(x) = sqrt ave$|f(x)|^2$
- Let Δk = rms deviation A(k)= sqrt ave$|A(k)|^2$
- Then $\Delta x \Delta k$ > or eq. ½

Examples:

- pulse width times bandwidth < or eq. K
- laser beam size times divergence < or eq. K
- Size source times number of eigenfunctions

Initial-Final Asymptotic Behavior

Horse Sense to check Answers

$$\left[sF(s)\right]_{s\to\infty} = f(t)_{t\to o^+} \qquad t^v \Rightarrow s^{v+1}$$

$$\left[sF(s)\right]_{s\to 0} = f(t)_{t\to\infty} \qquad x^v \Rightarrow k^{v+1}$$

- Laplace Transforms have the above asymptotic behaviors
- Fourier Transforms and Fourier Series in space have similar asymptotic behaviors.

Examples:

- FT or FS step (v = 0) has spectrum 1/k or 1/n
- FT of FS linear function (v = 1) has spectrum $1/k^2$ or $1/n^2$
- FT or FS delta function is constant

Edge and Corner Conditions

Horse Sense to check Answers

Jackson 2.11 3.4

- Derived from separation of variables in cylindrical and spherical coordinates
- Edge with open angle β in rad. => $\rho^{(\pi/\beta-1)}$
  - 90 degree open β= π/2 => $\rho^{(0.5)}$
  - 270 degree open β= 3π/2 => $\rho^{(-0.33)}$
  - 360 degree open β= 2π => $\rho^{(-0.5)}$
- Conical hole or sharp point $r^{(\nu-1)}$ data Fig. 3.6
  - Low fields in holes
  - Small tips ν = 0.2 to 0.1

$$\nu \cong [2\ln(\frac{2}{\pi-\beta}0]^{-1}$$

Summation of Complex Series

Horse Sense to check Answers

Jackson 2.10

$$z = complex\_number$$

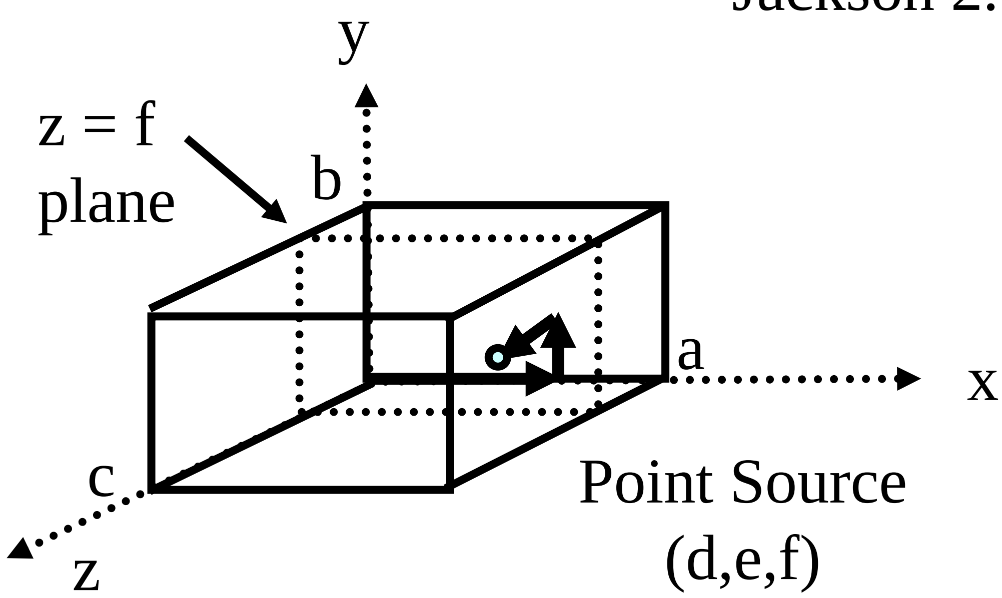
$$\sum_{n=0}^{\infty} z^n = 1/(1-z)$$

$$\sum_{n=0}^{\infty} \frac{z^n}{n} = -\ln(1-z)$$

- Fourier Transforms/Series, multiple reflections in electrodynamics, etc. lead to many complex expansions that can be summed up in closed form when estimating values.
- Be careful near z = 1

Separation of Variables: Geometry

Jackson 2.9



- Interior of a grounded conducting box bounded by planes of x = o, y = 0, z = 0, x = a, y = b, and z = c
- Point charge q at (d,e,f)

$$\nabla^2\psi = -\frac{4\pi q}{\varepsilon_0}\delta(x-d)\delta(y-e)\delta(z-f)$$

Separation of Variables: Product

Jackson 2.9

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(x, y, z) = X(x)Y(y)Z(z)$$

$$\frac{1}{X(x)}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z(z)}\frac{\partial^2 Z}{\partial z^2} = 0$$

- Method for solving differential equations by forming products that depend on one variable only and summing over all possible combinations of functions.
- Key Argument: Each term contains a function of one variable only and to hold for arbitrary values of all three variables each term must be constant

Separation of Variables: Eigenvalues

$$\frac{1}{X(x)}\frac{\partial^2 X}{\partial x^2} = -\alpha^2$$

$$\frac{1}{Y(y)}\frac{\partial^2 Y}{\partial y^2} = -\beta^2$$

$$\frac{1}{Z(z)}\frac{\partial^2 Z}{\partial z^2} = \gamma^2$$

$$\alpha^2 + \beta^2 = \gamma^2$$

$$\gamma_{nm} = \sqrt{\alpha_n{}^2 + \beta_m{}^2}$$

- Two oscillator (sin) and one exponentially damped (sinh)
- Boundary condition constraint gives discrete values of $\alpha_n = n\pi/a$ and $\beta_m = m\pi/b$.
- Then γ_{nm} picks up the slack to satisf PDE
- Two BC in z give Z(z)

$$A_{nm}\sinh(\gamma_{nm}z) + A'_{nm}\cosh(\gamma_{nm}z)$$

Separation of Variables: Representation

Jackson 2.9

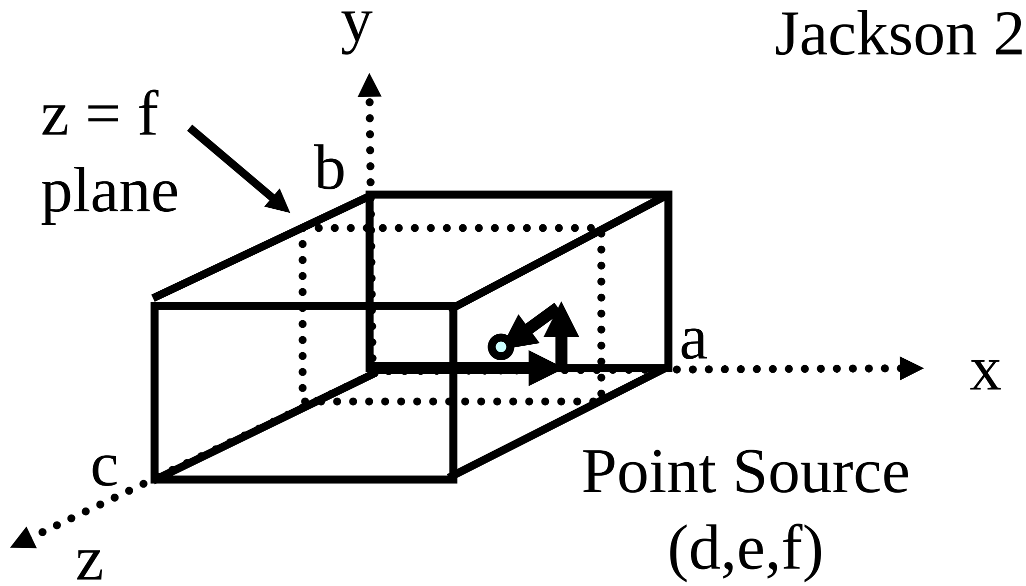
$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)$$

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} B_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm}(c-z))$$

- Sum is over eigenvalues in N-1 dimensions
- Note that the boundary conditions for x=0, x = a and for y=0 and y= b are met by sin behavior
- Note that the boundary conditions at z = 0 and z = c have already been applied in sinh behavior.

Separation of Variables: Source Strategy

Jackson 2.9



- View source as being on z = f plane.
- Require Φ_2 - Φ_1 =D(x,y)/ε_0 at z = f
- Also require at z =f $(\bar{E}_2 - \bar{E}_1) \cdot \hat{n} = \sigma_{SURFACE}(x, y) / \varepsilon_0$
- Multiply each of these equations by one of the composite eigenfunctions and integrate over x,y cross-section
- Gives two equations relating A_{nm} and B_{nm} for the same nm.

Separation of Variables: Source Results

$$A_{nm}\sinh(\gamma_{nm} f) + B_{nm}\sinh(\gamma_{nm}(c-f)) = 0$$

$$A_{nm}\cosh(\gamma_{nm} f) - B_{nm}\cosh(\gamma_{nm}(c-f)) = \sigma_{nm}$$

$$\sigma_{nm} = \frac{4}{ab}\int_0^a\int_0^b \sigma(x,y)\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)dxdy$$

$$\sigma(x,y) = -\frac{4\pi q}{\varepsilon_0}\delta(x-d)\delta(y-e)$$

Jackson 3.12

$$\sigma_{nm} = \frac{q16\pi}{\varepsilon_0 ab}\sin\left(\frac{2\pi d}{a}\right)\sin\left(\frac{2\pi e}{a}\right)$$

- The delta function source makes the source integral and expansion trivial

Separation of Variables: Final Result

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)$$

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} B_{nm} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} (c-z))$$

Jackson 3.12

- Solve for A_{nm} and B_{nm} and plug in
- Both proportional to σ_{nm}
- Also involve ratios of sinh and cosh
- See 3.168 pp 129