EE243 Advanced Electromagnetic Theory

Lec #3: Electrostatics (Apps., Form),

- **Electrostatic Boundary Conditions**
- **Energy, Force and Capacitance**
- **Electrostatic Boundary Conditions on Φ**
- **Image Solutions Example Green's Functions**
- **Integral Formulation**

Reading: Jackson
1.11, 2.1-2.5, 1.7-1.10

Electrostatic Boundary Conditions

- Div D = ρ

$$(E_2 - E_1) \cdot \hat{n} = \left(\frac{\delta \Phi_2}{\delta n} - \frac{\delta \Phi_1}{\delta n} \right) = \sigma / \varepsilon_0$$

D terminates on surface charge on a conductor

$d\Phi/dn = \sigma/\varepsilon_0$

- How about for Φ?
  - Jackson 1.6 evaluates dipole layer D(x)

$$(\Phi_2 - \Phi_1) \cdot \hat{n} = D / \varepsilon_0$$

  - Thus Φ is continuous unless there is a surface dipole layer

Energy

Jackson 1.11

- Electrostatic potential is potential energy of a charge
- Add a charge to m-1 charges = m-1 terms
- Repeat to add more charges (leaving out self-interactions) to get N charges
- Put in symmetric form (un-nest do loops to get ½ of regular double sum)

$$W = \frac{1}{2}\int \rho(\bar{x})\Phi(\bar{x})d^3x$$

Energy (Cont.)

$$W = \frac{1}{2}\int \rho(\overline{x})\Phi(\overline{x})d^3x$$

Use Poisson's Equation
Integrate by parts
Rewrite as E field

$$W = \frac{-\varepsilon_0}{2}\int [\nabla^2\Phi(\overline{x})]\Phi(\overline{x})d^3x = \frac{\varepsilon_0}{2}\int |\nabla\Phi(\overline{x})|^2\, d^3x = \frac{\varepsilon_0}{2}\int |\overline{E}(\overline{x})|^2\, d^3x$$

Physical interpretation: The electrostatic energy is stored in space as (1/2)**DE** and there is stored energy any time that the electric field is non-zero.

Force

- Calculated from change in energy for a small virtual displacement $\Delta W = F\ \Delta x$.
- Force per unit area Δa due to surface charge

$$w = \frac{\varepsilon_0}{2} |\,\overline{E}\,|^2 = \frac{\sigma^2}{2\varepsilon_0}$$

- Volume $\Delta a \Delta x$

$$\Delta W = -\frac{\sigma^2}{2\varepsilon_0} \Delta a \Delta x$$

- Outward force per unit area

$$F = \frac{\sigma^2}{2\varepsilon_0}$$

Capacitance

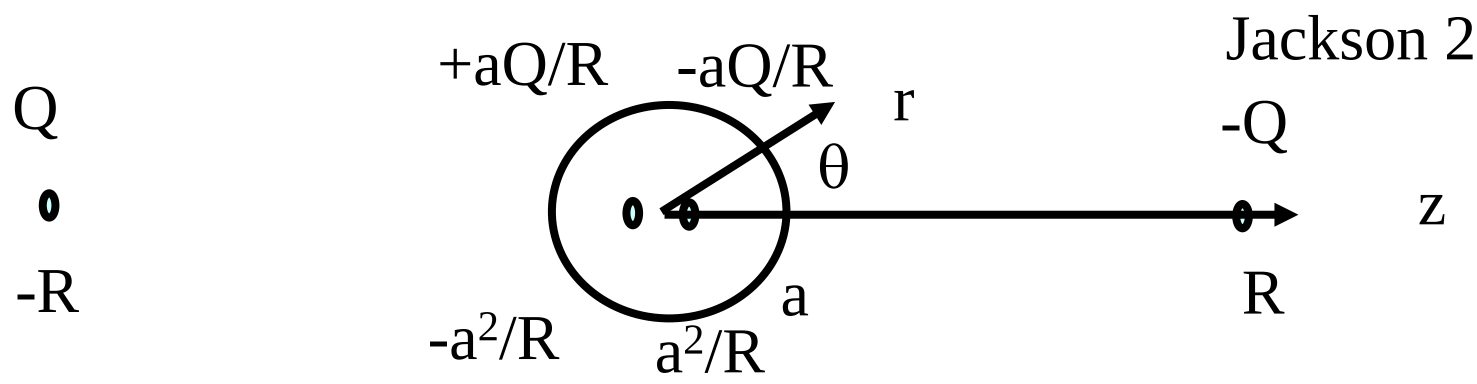
- Capacitance is defined as the charge per unit voltage when all other conductors are grounded
- Mutual capacitance is charge per unit voltage difference when a pair have equal and opposite charge and all other conductors are grounded
- Potential is sum over charges
- Potential Energy found by adding new potential to m-1 => half double sum $(1/2)C_{ij}V_iV_j$

Method of Images

Jackson 2.1-2.4

- Under favorable (and rare) conditions inferred from a geometry a small number of external charges can simulate the required boundary conditions.
- Examples for Dirichlet (G = 0 on boundary)
  - Charge above a conducting plane
    - Charge -q at position -y
  - Charge in a 360/n wedge
  - Charge outside a conducting sphere
    - Charge -aQ/y at $y' = a^2/y$
  - Charge inside a spherical hole in a conductor
- Examples of Neumann = Are there any?

Conducting Sphere in a Uniform E field



- Consider two charges (to create uniform field in limit R => infinity and Q/R^2 constant)
  - -Q at y = R and +Q at y = -R
- Add images to make G = 0
  - +aQ/R at $+a^2/R$ and –aQ/R at $–a^2/R$
- Potential is 4 terms
- Assume R >> a; use $1/(1+x)^{1/2}$ approx. 1-x
- Take limit R => infinity and Q/R^2 constant

Conducting Sphere - Uniform E field (Cont.)

Potential

$$\Phi = -E_0 (r - \frac{a^3}{r^2}) \cos\theta$$

Physically interpret as dipole (charge times separation)

$$D = \frac{Qa}{R}\frac{2a^2}{R} = 4\pi\varepsilon_0 E_0 a^3 = 3\varepsilon_0 E_0 \cdot Volume$$

D is 3D times volume and is oriented directly opposite to the applied field

Surface charge density (from D normal) is 3D

$$\sigma_{surface} = -\varepsilon_0 \frac{\delta\Phi}{\delta r}|_{r=a} = 3\varepsilon_0 E_0 \cos\theta$$

Green's Theorem and Integral

Green's 2nd Identity (Theorem)

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)d^3x = \oint_{\delta V}\left[\phi\frac{\delta\psi}{\delta n} - \psi\frac{\delta\phi}{\delta n}\right]da$$

Use ϕ = Φ and Poisson's Equation for F

Use ψ = G any solution to Poisson's Equation for one point charge in the internal region and any boundary conditions on dV

$$\nabla'^2\Phi(\bar{x},\bar{x}') = -\frac{\rho(\bar{x})}{\varepsilon_0} \qquad \nabla'^2\Psi(\bar{x},\bar{x}') = -4\pi\delta(\bar{x}-\bar{x}')$$

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')G(\bar{x},\bar{x}')d^3x' + \frac{1}{4\pi}\oint_S\left[G(\bar{x},\bar{x}')\frac{\delta\Phi}{\delta n'} - \Phi(\bar{x}')\frac{\delta G(\bar{x},\bar{x}')}{\delta n'}\right]da'$$

Common Case: Integral Representation with the Free Space Green's Function

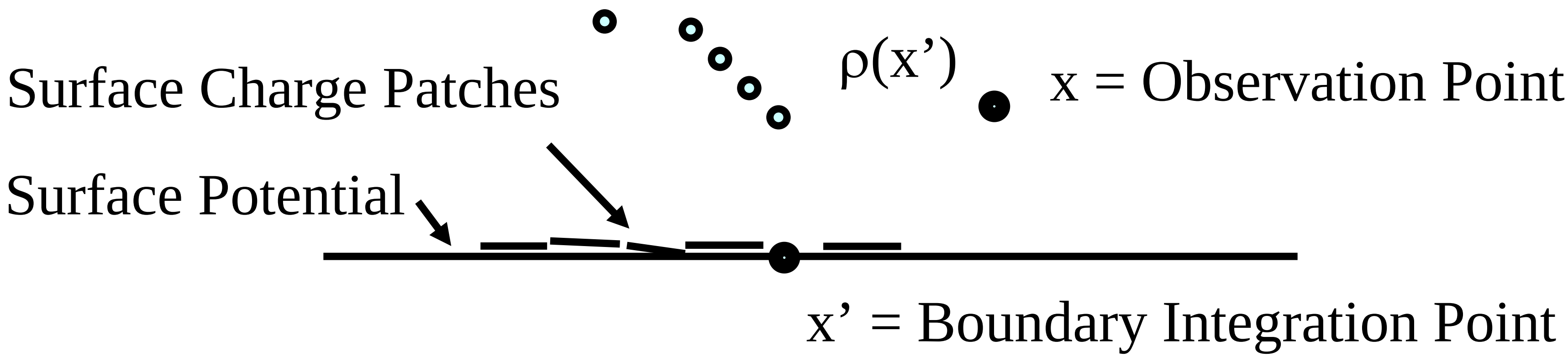
For a unit charge in free space the potential is proportional to

$$G(\bar{x}, \bar{x}') = \frac{1}{|\bar{x} - \bar{x}'|}$$

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\bar{x}') \frac{1}{|\bar{x} - \bar{x}'|} d^3x' + \frac{1}{4\pi} \oint_S \left[\frac{1}{|\bar{x} - \bar{x}'|} \frac{\delta\Phi}{\delta n'} - \Phi(\bar{x}') \frac{\delta}{\delta n'} \left(\frac{1}{|\bar{x} - \bar{x}'|} \right) \right] da'$$

Need to know:
1) Charge distribution in interior
2) The potential on the boundary
3) The derivative of the potential normal to the boundary on the boundary (surface charge)

Example Green's Function Application



$$\Phi(\overline{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\overline{x}')\frac{1}{|\overline{x}-\overline{x}'|}d^3x' + \frac{1}{4\pi}\oint_S\left[\frac{1}{|\overline{x}-\overline{x}'|}\frac{\delta\Phi}{\delta n'} - \Phi(\overline{x}')\frac{\delta}{\delta n'}\left(\frac{1}{|\overline{x}-\overline{x}'|}\right)\right]da'$$

- Observation point is in solution region
- Surface integration points are on boundary
- Volume integration is over solution region