

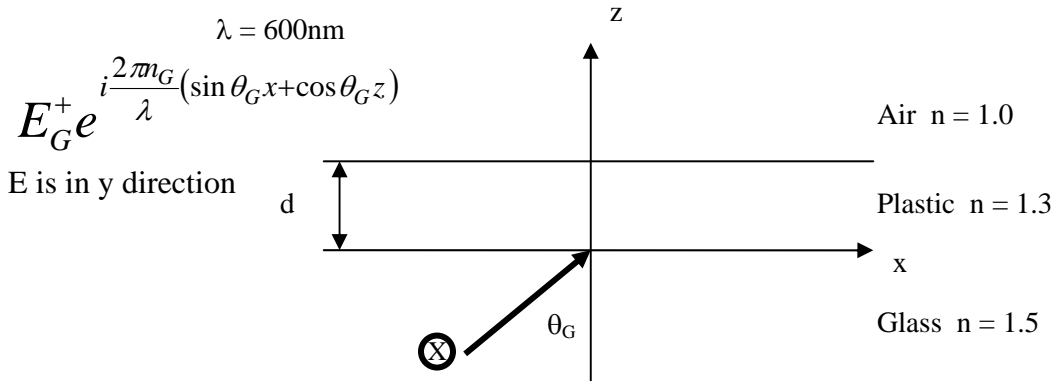
**EECS 210**  
 Fall 2006  
 Tu, Th 12:30-2  
 400 Cory

**Applied Electromagnetic Theory**  
 Office Hours  
 M, (W), 11AM  
 Tu, Th, (F) 10AM

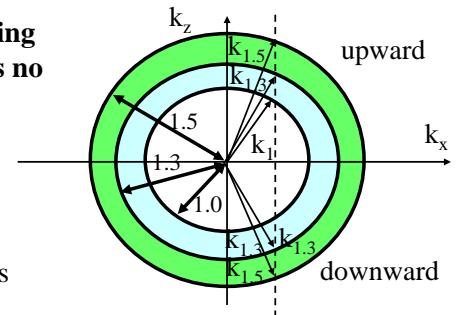
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## Homework # 6: Solution



**Note:** There are multiple approaches for solving this problem. Using transverse component as illustrated here works well when there is no variation in the 3<sup>rd</sup> dimension. Other approaches are the  $z$  components (Jackson) and  $z$  components of the vector potential (Harrington).



### 6.1) Kinetic Boundary Conditions and k-vectors:

- Start a  $k$ -vector plot by drawing the  $k_x$  and  $k_z$  axes and concentric circles of radius 1, 1.3 and 1.5. (The 1 here indicates that the  $k$ -vectors are normalized to the  $k$  for a plane wave in free space.)
- Sketch the 5  $k$ -vectors for the plane waves that will arise for an incidence angle of  $30^\circ$  in glass.
- Compute the angles in plastic and air and show that the  $k$ -vector  $x$ -component is the same for all 5  $k$ -vectors.

Using Snell's law for the 3 media gives  $30, 35.2$  and  $48.6$  degrees.

- Find the period of the variation parallel to the surface. (It is larger than  $600\text{ nm}$  and this allows radiation in air).

In general it is easiest to use  $\lambda_{\text{surface}} = 2\pi/k_x$  where  $k_x = k_0 n \sin \theta = (2\pi/\lambda) 1.5 \sin 30 = (2\pi/\lambda) 0.75$ .

$\lambda_{\text{surface}} = 2\pi/k_x = \lambda_{\text{air}} / 0.75 = 1.333 \lambda_{\text{air}} = 800\text{nm}$ . The  $k_z$  values follow from the wave equation

$$\text{constraint } k_{zi} = \sqrt{k_0^2 n_i^2 - (k_0 n_G \sin \theta_G)^2} = \frac{2\pi}{\lambda_{\text{air}}} \sqrt{n_i^2 - (n_G \sin \theta_G)^2} = \frac{2\pi}{\lambda_{\text{air}}} \sqrt{n_i^2 - (0.75)^2} \text{ This}$$

gives  $(2\pi/\lambda_{\text{air}})$  times 1.299, 1.06 and 0.661.

- Find the  $k$ -vectors for an incidence angle of  $60^\circ$  in glass find the period of variation along the surface. (It is smaller than  $600\text{nm}$  and no radiation occurs.)

Here  $1.5 \sin 60 = 1.30$ . This makes the angle in the plastic  $90$  degrees and the angle in air imaginary. The  $k_z$  values can be found from the wave equation constraint even when the angle is

$$\text{imaginary. } k_{zi} = \sqrt{k_0^2 n_i^2 - (k_0 n_G \sin \theta_G)^2} = \frac{2\pi}{\lambda_{\text{air}}} \sqrt{n_i^2 - (n_G \sin \theta_G)^2} = \frac{2\pi}{\lambda_{\text{air}}} \sqrt{n_i^2 - (1.30)^2} \text{ This}$$

gives  $(2\pi/\lambda_{\text{air}})$  times 0.75, 0, and  $i0.83$  (so decay away from surface in air).

### 6.2) Dynamic Boundary Conditions:

a) Write out phasor expressions for the plane wave fields as a function of x and z for each of the 5 k-vectors in terms of a complex constant in each region using the example notation above.

$$E_{ay}^+(\bar{x}) = E_{a0}^+ \hat{y} e^{+ik_z z} e^{+ik_x x}$$

$$E_{py}(\bar{x}) = E_{p0}^+ \hat{y} e^{+ik_z z} e^{+ik_x x} + E_{p0}^- \hat{y} e^{-ik_z z} e^{+ik_x x}$$

$$E_{gy}^-(\bar{x}) = E_{g0}^+ \hat{y} e^{+ik_z z} e^{+ik_x x} + E_{g0}^- \hat{y} e^{-ik_z z} e^{+ik_x x}$$

b) Write sufficient boundary conditions at z = 0, plug in the fields and evaluate derivatives. Two boundary conditions are needed at z = 0. Ey continuous and Hx continuous.

$$E_{p0}^+ e^{+ik_x x} + E_{p0}^- e^{+ik_x x} = E_{g0}^+ e^{+ik_x x} + E_{g0}^- e^{+ik_x x}$$

$$H_x = \frac{-1}{i\omega\mu} \frac{\partial E_{iy}}{\partial z}$$

$$k_{zp} E_{p0}^+ e^{+ik_x x} - k_{zp} E_{p0}^- e^{+ik_x x} = k_{zg} E_{g0}^+ e^{+ik_x x} - k_{zg} E_{g0}^- e^{+ik_x x}$$

c) Write sufficient boundary conditions at z = d, plug in the field and evaluate derivatives.

$$E_{p0}^+ e^{ik_z d} e^{+ik_x x} + E_{p0}^- e^{-ik_z d} e^{+ik_x x} = E_{a0}^+ e^{ik_z d} e^{+ik_x x}$$

$$H_x = \frac{-1}{i\omega\mu} \frac{\partial E_{iy}}{\partial z}$$

$$k_{zp} E_{p0}^+ e^{ik_z d} e^{+ik_x x} - k_{zp} E_{p0}^- e^{-ik_z d} e^{+ik_x x} = k_{za} E_{a0}^+ e^{ik_z d} e^{+ik_x x}$$

d) Show that the kinematic condition can be factored out of these boundary conditions. All four boundary conditions above have the same z variation which can be factored out.

### 6.3) Physical Effects:

a) Find the time-average and instantaneous Poynting vector in air for 30° incidence in glass. Assume  $E_{a0}^+$  is known.

$$\bar{E}_a = E_{a0}^+ e^{ik_z z} e^{+ik_x x} \hat{y}$$

$$\bar{H}_a = \frac{-1}{i\omega\mu} \frac{\partial E_{ay}}{\partial z} \hat{x} + \frac{1}{i\omega\mu} \frac{\partial E_{ay}}{\partial x} \hat{z} = -\frac{ik_z}{i\omega\mu} E_{ay} \hat{x} + \frac{ik_x}{i\omega\mu} E_{ay} \hat{z}$$

$$\bar{S}_{ave} = \frac{1}{2} (\bar{E} \times \bar{H}^*) = \frac{1}{2\omega\mu} E_{a0}^+ E_{a0}^{+*} (k_x \hat{x} + k_z \hat{z})$$

$$\bar{E}_a(\bar{x}, t) = |E_{a0}^+| \cos(k_x x + k_z z + \phi_a - \omega t)$$

$$\bar{H}_a(\bar{x}, t) = \frac{|E_{a0}^+|}{\omega\mu} (-k_z \hat{x} + k_x \hat{z}) \cos(k_x x + k_z z + \phi_a - \omega t)$$

$$\bar{S} = \frac{|E_{a0}^+|^2}{\omega\mu} (k_x \hat{x} + k_z \hat{z})$$

b) Find the time-average and instantaneous Poynting vector in air for 50° incidence in glass. See above.

c) Show that for 50° incidence in glass that the z-component of the Poynting vector is **imaginary** for time average and **2X frequency** for time-varying.

$$\bar{E}_a = E_{a0}^+ e^{-v_z z} e^{+ik_x x} \hat{y}$$

$$\bar{H}_a = \frac{-1}{i\omega\mu} \frac{\partial E_{ay}}{\partial z} \hat{x} + \frac{1}{i\omega\mu} \frac{\partial E_{ay}}{\partial x} \hat{z} = \frac{+v_{az}}{i\omega\mu} E_{ay} \hat{x} + \frac{ik_x}{i\omega\mu} E_{ay} \hat{z}$$

$$\bar{S}_{ave} = \frac{1}{2} (\bar{E} \times \bar{H}^*) = \frac{1}{2\omega\mu} E_{a0}^+ E_{a0}^{+*} (k_x \hat{x} + iv_{az} \hat{z})$$

$$\bar{E}_a(\bar{x}, t) = |E_{a0}^+| e^{-v_{az} z} \cos(k_x x + \phi_a - \omega t)$$

$$\bar{H}_a(\bar{x}, t) = \frac{|E_{a0}^+|}{\omega\mu} e^{-v_{az} z} \left( -k_z \hat{x} \cos(k_x x + \phi_a - \omega t) + v_z \hat{z} \cos\left(\frac{\pi}{2} + k_x x + \phi_a - \omega t\right) \right)$$

$$\bar{S}_{inst} = \frac{|E_{a0}^+|^2}{\omega\mu} (e^{-v_{az} z})^2 \left\{ k_x \hat{x} [\cos(k_x x + \phi_a - \omega t)]^2 - v_z \hat{z} \left[ \cos(k_x x + \phi_a - \omega t) \cos\left(\frac{\pi}{2} + k_x x + \phi_a - \omega t\right) \right] \right\}$$

d) Define the transverse impedance as  $E_y/H_x$  and show that this impedance is real for  $30^\circ$  incidence in glass and becomes a capacitive reactance for  $50^\circ$  incidence in glass.  
(Sign below does not seem correct as it looks inductive => error somewhere)

$$H_{ax} = \frac{-1}{i\omega\mu} \frac{\partial E_{iy}}{\partial z} = \frac{-ik_z}{i\omega\mu} E_{ay}$$

$$Z_{+z} = \frac{E_{ay}}{-H_{ax}} = \frac{\omega\mu}{k_z} = \text{Real}$$

$$H_{ax} = \frac{-1}{i\omega\mu} \frac{\partial E_{ay}}{\partial z} = \frac{+v_z}{i\omega\mu} E_{ay}$$

$$Z_{+z} = \frac{E_{ay}}{-H_{ax}} = \frac{-i\omega\mu}{v_z} = \text{Im } y$$

e) Show that the ratio of the incident and reflected waves in plastic at d is independent of d itself and hence the standing wave ratio in plastic does not depend on thickness.

In 6.2b the upper and lower equations can be weighted appropriately and subtracted to eliminate the amplitude in air. Since no constant term appears it is then possible to divide by the upward traveling amplitude in plastic and solve for the ratio of the downward amplitude over the upward amplitude. Specifically, it is necessary to solve for the amplitude ration at  $z = d$  and it should agree with the single surface reflection formula 7.39 pp. 305 Jackson

$$\rho = \frac{E_{p0}^- e^{-ik_p d}}{E_{p0}^+ e^{ik_p d}} = \text{Eq 7.39}$$