

**EECS 210**  
**Fall 2006**  
**Tu, Th 12:30-2**  
**400 Cory**

**Applied Electromagnetic Theory**  
**Office Hours**      **Prof. A. R. Neureuther,**  
**M, (W), 11AM**      **509 Cory Hall, 2-4590**  
**Tu, Th, (F) 10AM**      **neureuth@eeecs**



## Homework # 5: Due 5 PM Monday Oct 9th

The goal of 5.1 and 5.2 are to illustrate how time varying signals must be coordinated in delay in time-domain and phase in frequency domain in a position dependent manner to coherently fully add together. Unfortunately, going to E and then H keeps generating a lot more algebra as both the numerator and denominator must be differentiated. To avoid this algebra both E and H were dropped at the last minute from the assignment. However if you did this algebra you are in good shape for computing the far fields from a radiating current source that comes up in a couple of weeks.

5.1) **Retarded Potential (time-domain):** Consider the scalar potential produced by two time-varying sources. One source is at  $x_1 = (0,0,0)$  and has charge  $q_1(t)$ , the other is at  $x_2 = (a,0,0)$  and has charge  $q_2(t)$ .

a) Using the scalar potential from a point charge write down a general expression for  $\Phi(x,y,z,t)$ . The answer is two terms like Eq. 6.44 of Jackson with the two source positions substituted, the time retardation based on position and delta replaced by  $q_1(\text{time delay } 1)$  and  $q_2(\text{time delay } 2)$ .

(The physical observation here is that to find the signal at  $x$  one needs to look at the sources at earlier times that are not the same but differ. They differ NOT by the delay for the signal from one source to reach the other source but by the difference in their relative distances to the observation point divided by the speed of light.)

b) Find the Electric field contributed by this potential  $E(x,y,z,t)$  far from the source in the  $z = 0$  plane.

Simplifying to potential, the relative delay between the sources is  $a/c$ .

c) Now suppose  $q_2(t) = q_1(t - \tau)$ . Find the value of  $\tau$  that for large positive distances on the  $x$ -axis will synchronize the contributions from the two sources regardless of the time-variation of  $q_1(t)$ . The synchronization is to delay source 2 by  $a/c$ .

d) Repeat part c) for a large distance in a direction  $n = (\cos\phi, \sin\phi, 0)$  i.e.  $(R\cos\phi, R\sin\phi, 0)$  where  $R \gg a$ .

For an arbitrary direction in the  $z = 0$  plane, the relative shorter distance to the source depends on the direction to the source. First neglect the effect of  $a$  in the denominator and just use  $R$  for both terms. Next approximate the length of the vector  $(R\cos\phi - a, R\sin\phi, 0)$  that appears in the delay using a Taylor series expansion as  $R(1 - 2aR\cos\phi)^{0.5} = R - a\cos\phi$ .

e) Show how the delay can be described through using  $(x_2 \cdot n)$ .

$(a, 0, 0) \cdot (1/R)(R\cos\phi, R\sin\phi, 0) = (1/R) aR\cos\phi = a\cos\phi = (x_2 \cdot n)$

f) Find  $E(R, 0, 0, t) \cdot E(R, 0, 0, t)$  where  $R \gg a$ , and show that it is given by  $E_1 E_1 + E_2 E_2$  plus a cross term proportional to  $E_1 E_2$  and involving the phase between the sources and  $x_2 \cdot n$ .

Again simplifying to potential and neglecting the distance effect in the denominator just gives  $(1/R)$  times  $[(q_1 \text{ delayed } R/c) \text{ plus } (2 \text{ delayed } (R/c - a/c))]$ . So in addition to a common large delay there is a relative time advance of the nearer source by  $a/c$ . Squaring the potential then gives  $q_1$  squared at time delayed  $R/c$  plus  $q_2$  squared at time delayed by  $R/c - a/c$ . and then a cross term involving 2 times the product of  $q_1$  at delay  $R/c$  and  $q_2$  at delay  $R/c - a/c$ . So the signals being cross multiplied were created at different time as occurs in the cross correlation of two functions of time.

The main point of this problem is that in evaluating the fields at a given time  $t$  it is necessary to take into account what a distributed source did in the past not at a single time  $t - R/c$  but at a past time that varies across the physical position of the source components.

5.2) **Retarded Potential (frequency-domain):** Now reconsider the two point time-varying source in Problem 5.1).

What happened in 5.1 in retarded time will now happen in 5.2 in retarded phase.

First an overview:

- Philosophically, one could consider time the fourth dimension and the factor  $e^{-i\omega t}$  as the eigenfunction for the 4<sup>th</sup> dimension which multiplies each of the spatial eigenfunctions is then summed in the fourth variable over all  $\omega$  (integrated because time is unbounded and gives a continuous distribution).
- In engineering we typically assume that multiplying by the eigenfunction and integrating is implicit and only show the integrand as a sum over spatial eigenfunctions with amplitude coefficients that are phasors that are frequency dependent such as  $\Phi(x,y,z,\omega)$ .
- Another useful observation is that a delay of a signal by a time  $\tau$ , can be shown by change of variables in integration as equivalent to multiplying by  $e^{-i\omega\tau}$
- Since  $q_1(t)$  and  $q_2(t)$  are real their Fourier representations in frequency space will satisfy  $Q_i(\omega) = Q_i^*(-\omega)$

a) Find an expression for  $\Phi(x,y,z,\omega)$ .

This is the Fourier transform of 5.1.a. Only the numerator depends on time but it is a delayed time that depends on position  $(x,y,z)$ . The Fourier transform of the numerator is thus  $Q_i(\omega)e^{+i\omega\tau_i}$  where  $\tau_i$  is  $|(x_i,y_i,z_i)|/c$

b) Find the Electric field contributed by this potential  $E(x,y,z,\omega)$  far from the source in the  $z = 0$  plane. Hint use  $\mathbf{n} = (\cos\phi, \sin\phi, 0)$ .

Again simplifying to potential, neglecting the effect on the denominator and assuming the observation point is in the  $z = 0$  plane gives  $Q_i(\omega)e^{-i\omega\tau_i}$  where  $\tau_1$  is  $|(R\cos\theta, R\sin\theta, 0)|/c$  and  $\tau_2$  is  $|(R\cos\theta - a, R\sin\theta, 0)|/c$

c) Using the curl E Maxwell equation find  $H(x,y,z,\omega)$  far from the source in the  $z = 0$  plane.

(skipped because of algebra on algebra where ME keeps adding terms to become self consistent).

d) Show that  $H(x,y,z,-\omega) = H^*(x,y,z,\omega)$

This frequency domain behavior is inherited by all field components and arises because  $\mathbf{q}_1(t)$  and  $\mathbf{q}_2(t)$  are real.

e) For the case of  $q_2(t) = q_1(t-\tau)$ , find  $q_2(\omega)$  in terms of  $q_1(\omega)$ .

$$Q_2(\omega) = Q_1(\omega)e^{-i\omega\tau}$$

f) Find the product  $E(R,0,0, \omega) \cdot E(R,0,0, \omega)$  where  $R \gg a$ , and show that it is given by  $E_1E_1 + E_2E_2$  plus a cross term proportional to  $E_1E_2$  and involving the phase between the sources and  $(\mathbf{x}_2 \cdot \mathbf{n})$ .

Apply to the product the potential with its conjugate to simplify both algebra and to time-average and assume the denominator is not affected by  $a$ .

Product =  $(1/R^2)[ Q_1(\omega) Q_1^*(\omega) + \{ Q_1(\omega) Q_2^*(\omega) e^{+i\omega\tau} + Q_1^*(\omega) Q_2(\omega) e^{-i\omega\tau} \} Q_2^*(\omega) ]$  where  $\tau = a/c$ . Here the cross term contains a phase delay or advance.

4.3) **Green's Function in Time-harmonic:** Consider the interior of a grounded box defined by the six planes,  $x = 0$ ,  $y = 0$ ,  $z = 0$ ,  $x = a$ ,  $y = b$ , and  $z = c$ . A time-varying charge source is given by  $q(x,y,z,t) = \delta(x-d)\delta(y-e)\delta(z-f)\delta(t-\tau)$ .

a) Convert this source to a Fourier representation using  $q(x,y,z,\omega)$

$$q(x,y,z,\omega) = \delta(x-d)\delta(y-e)\delta(z-f) e^{-i\omega\tau}$$

b) Use the N-dimensional eigenfunctions and the scalar wave equation to find the solution for the potential inside the box.

$$\Phi(x, y, z, \omega) = \frac{32}{\pi abc} \sum_{l,m,n=1} e^{-i\omega t} \frac{\sin(\frac{l\pi x}{a}) \sin(\frac{l\pi y}{a}) \sin(\frac{m\pi y}{b}) \sin(\frac{m\pi z}{b}) \sin(\frac{n\pi z}{c}) \sin(\frac{n\pi t}{c})}{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} - \omega^2 \mu \epsilon}$$

Three important things.

- 1) Each coefficient is now a phasor with both magnitude (same as in statics) and phase (which is nonzero due to the delay).
- 2) The wave equation replaced Laplace's equation by adding the second derivative with respect to time.
- 3) This time differentiation results in the term  $-\omega^2 \mu \epsilon$  that subtracts from the eigenvalues in the denominator.

c) Describe what happens to the potential when the time-harmonic frequency contribution  $\omega^2 \mu \epsilon$  hits an eigenvalue.

When this negative term equals the eigenvalue the potential blows up. This is a resonance effect.

c) Suppose instead you had used the N-1 eigenfunction expansion method what would happen when  $\omega^2 \mu \epsilon$  hits an eigenvalue and then increases further?

In this approach the functional variation in the 3<sup>rd</sup> Dimension is chosen to meet the constraint of the wave equation. In statics an exponential decay/growth is always required. Here when

$\frac{l^2}{a^2} + \frac{m^2}{b^2} < \omega^2 \mu \epsilon$  the behavior in the z direction can become oscillatory (A sin + B cos) instead of (A sinh + B sinh). Now the N-1 solution in Equation 3.168 of Jackson contains a Sinh. With the

above changes this sinh becomes a sin. The Klm becomes  $\frac{l^2}{a^2} + \frac{m^2}{b^2} - \omega^2 \mu \epsilon$  and when it equals

$\frac{n^2}{c^2}$  the sin in the denominator goes to zero and the field blows up.

Buzz Lighyear sez "To infinity and beyond."