Version 10/12/06

EECS 210 Applied Electromagnetic Theory

Fall 2006 | Office Hours | Prof. A. R. Neureuther,
Tu, Th 12:30-2 | M, (W), 11AM | 509 Cory Hall, 2-4590
400 Cory | Tu, Th, (F) 10AM | neureuth@eecs

Homework # 4: Solution

4.1) **Boundary Value Problem N-1 Technique**: Consider the interior of a grounded box defined by the six planes, x = 0, y = 0, c = 0, x = a, y = b, and c = z. A surface charge distribution is on the plane z = f and has a constant value $\sigma_{SURFACE}$ for a/2 –d <x < a/2 +d. Note that this source is not a function of y.

a) Write down a general expression using eigenfunctions in x and y and exponentials in z that can represent the potential inside the box for z < f and for z > f for any source on the z = f plane in the grounded box (N-1 technique).

$$\Phi_<(x,y,z) = \sum_{l,m} A_{l,m} \sin(\frac{l\pi x}{a})\sin(\frac{m\pi y}{b})\sinh(\gamma_{l,m} z)$$

$$\Phi_>(x,y,z) = \sum_{l,m} B_{l,m} \sin(\frac{l\pi x}{a})\sin(\frac{m\pi y}{b})\sinh(\gamma_{l,m} c - z)$$

b) Specify and apply the boundary condition on the potential at the source. Use the orthogonal nature of the expansion to show that this BC applies to each individual term in the expression.

$\Phi_<(x,y,f) = \Phi_>(x,y,f)$ so multiply by an eigenfunction and integrate over the (x,y) cross section. This picks out one term from the sum on each side giving

$$A_{l,m}\sinh(\gamma_{l,m} f) = B_{l,m}\sinh(\gamma_{l,m}(c-f))$$

c) Specify and apply the boundary condition on the normal derivative of the potential at the source. Use the orthogonal nature of the expansion to in effect expand the source on the source plane and apply the BC to each term.

$D_{z>} - D_{z<} = \sigma_{source}$ Get D from –εGradΦ in z direction. Then multiple by eigenvalue and integrate over x,y cross section.

$-\varepsilon A_{l,m}\gamma_{l,m} - \varepsilon B_{l,m}\gamma_{l,m} = \sigma_{l,m}$ Here m = even drops out as area integrates to zero. Coefficient for m = odd is inversely proportional to m. In x the sin function integrates to cos and then gets evaluated at x = a/2-d and x = a/2 +d. Although it is hard to this is similar to integrating over a finite pulse width which when centered a x = 0 would give sin u/u where u is proportional to 1/d. is amount to

$$\sigma_{l,m} = \frac{\sin\left(\frac{(\ )}{d}\right)}{\frac{(\ )}{d}}\frac{1}{m}$$

d) Solve for the coefficients and write out the final N-1 infinite summation for the potential for z < f.

Answer is similar to 3.168 in Jackson but with

- even terms in y direction removed, m = odd terms in y direction weighted 1/m,
- both l = odd and l = even terms in x direction and have hidden sinu/u .
- (Also 2ε from left divides right to get A once B is eliminated B=A).

e) Comment on the nature of the solution. (In which dimension is it constant? How did (sin(u))/u appear?)
The solution is constant in the y direction. There is a hidden sinu/u that comes from integrating over the less than full width (support) of the charge in the x direction.

4.2) **Surface Boundary Value Problem N Technique**: Now reconsider the grounded box and source in Problem 4.1).
a) Write down an expansion for the potential inside the box using eigenfunctions in all three directions (N technique). Use A_{ijk} as the unknown coefficient for the composite eigenfunction made up of the product of the eigenfunction in x times the eigenfunction in y times the eigenfunction in z.
Jackson 3.166
b) Substitute this expansion in to Poisson's Equation for the potential inside the box.
When substituted the Laplacian operator can be carried out to give the eigenvalue.
c) Multiply both sides by the i,j,k composite eigenfunction and integrate over the volume of the box to determine a value for A_{ijk}.
This process picks out one term from the expansion on the differential equation side. The integration over the source then determines the relative weight of the eigenfunction used in the source. Dividing by the normalization integral from the left side then gives the coefficient in the expansion.
d) Write out the solution for the potential as the triple infinite summation over the eigenvalues.
Very similar to Jackson 3.167.

- The z behavior is a delta function so sin sin stays in z.
- The m = even coefficients are zero and the m = odd contain a weighting 1/m.
- The l = odd and even coefficients are not zero zero and even follow shifted sinu/u wher u is inversely proportional to a.

d) Comment on the nature of the solution. (In which dimension is it constant? How did (sin(u))/u appear?
Constant in y. Shifted sinu/u from expansion of symmetric function with finite support.

4.3) **Dielectric Polarization**: Consider a charge q at a distance **a** along the z axis from a dielectric plane at z = 0. Use the image charge solution from Jackson.
a) Evaluate the potential as a function of radius on z = 0. (Is constant or non constant?).
There are two sources for z > 0 (an image and the original). Then one source for z <0 as shown in Jackson pp 156. Using either expression near z = 0 boundary shows that the potential is clearly non-zero whenever the permittivity is not infinite. (This means in an integral equation the integration over the potential will be required with most Green's functions.)
b) Derive and expression for the polarization in the dielectric.
This is $P = -(\varepsilon-\varepsilon_0)E$ throughout the volume of the dielectric
c) Find the polarization charge on the surface of the dielectric as a function of radius from the divergence of P.
This is Jackson 4.46 and 4.47.
Note this is not real charge density that makes D normal discontinuous. Rather it is a charge density that can be viewed as generating an electric field in the opposite direction that lowers the total E inside the dielectric region. See Figure 4.7 in Jackson.