EECS 210   Applied Electromagnetic Theory

Fall 2006   **Office Hours**   **Prof. A. R. Neureuther,**
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Homework # 2: Solution

2.1) **Surface charge singularity contribution**: Consider the potential due a continuous distribution of charge on a smooth surface as an observation point $\bar{x}_o^{\,'}$. that is ε away from the surface approaches an observation point $\bar{x}_0$ on the surface. Consider a small disk in the surface centered on $\bar{x}_0$ of radius **a** over which the charge density can be approximated as constant. Break up the integral for the potential at $\mathbf{x}_0$ into a part outside the circle and a part over the circle. Show that when ε << a the integral for the electric field (written out as a function of $\left|\vec{x}_{surface} - \vec{x}_0^1\right|$) over the circular disk integrates to a factor of σ /ε₀.

Solution Comment: (This problem originally was to calculate the potential which should be continuous in crossing the sheet of charge. The 1/(distance) integral, however, is very problematic as the integrand becomes constant at large distances and the integral blows up. This problem is apparently associated with the definition of potential in which only the change in potential is physical.)

Solution for Electric field normal to the screen: The integrand has a 1/(distance)^2 and a cosθ to pick out the component perpendicular to the screen toward (0,0,ε). But cosθ is really -ε/distance. This gives

$$E(0,0,\varepsilon) = \frac{1}{4\pi\varepsilon_0} \int_0^\infty \int_0^{2\pi} \frac{-\varepsilon\sigma_{SURFACE}}{\left(\varepsilon^2 + \rho^2\right)^{3/2}} \rho\delta\rho = \frac{\sigma_{SURFACE}}{4\pi\varepsilon_0} \left. \frac{-2\pi\varepsilon}{\left(\varepsilon^2 + \rho^2\right)^{1/2}} \right]_0^\infty = \frac{-\sigma_{SURFACE}}{2\varepsilon_0}$$

2.1) **Surface Charge Density, Force, Capacitance and Stored Energy**: Consider a charge q at a distance **a** along the z axis from a conducting plane at z = 0. Use the image charge solution to help make calculations.

a) Find the surface charge density on the conducting plane as a function of the radial distance in the z = 0 plane.

As in 2.1 this is

$$\sigma_{SURFACE} = D_z = \varepsilon_0 E_z(\rho,\phi,0) = \frac{1}{4\pi_0} \frac{-aq}{\left(a^2 + \rho^2\right)^{3/2}} + \frac{a(-q)}{\left(a^2 + \rho^2\right)^{3/2}} = \frac{-q}{2\pi} \frac{a}{\left(a^2 + \rho^2\right)^{3/2}}$$

b) Show that there is no force parallel to the surface of the conductor.
A force parallel to the surface would require a D and hence E parallel to the surface. Since CurlE =0 and there is no E inside the conductor outside there can be no E or D parallel to the conductor.

c) Show that the net force perpendicular to the conductor adds up in magnitude to the force on the charge q.
This is the same integral from 2.1 and integrand inside gives 2π and total of –q.

d) Assume that the charge q sits on a sphere of radius a/1000 so that the voltage is defined and the fields on the plane do not change much. Evaluate the mutual capacitance to the ground plane. Be sure to eliminate the capacitance of the sphere to infinity without the ground plane.

$$\Phi_{SPHERE} = \frac{q}{4\pi\varepsilon_0(a/1000)} - \frac{q}{4\pi\varepsilon_0(2a)}$$

$$C_{TOTAL} = \frac{q}{\Phi_{SPHERE}} = \frac{4\pi\varepsilon_0 a}{1000 - 0.5} = \frac{4\pi\varepsilon_0 a}{1000} + \frac{4\pi\varepsilon_0 a}{2x10^6} = C_{SELF} + C_{IMAGE}$$

e) Using the point source, find the stored energy by finding the total energy and eliminating the self-energy.

The challenge here is that the charge and image charge are linked together and cannot be moved independently. So instead consider the energy in the half space $z > 0$. This is half the energy if q is present and $-q$ is brought up $= (1/2)$ q $(q/4\pi 2a) = q^2/8\pi a$.

2.3) **Equivalence Theorem**: Consider a charge q at a distance **a** along the z axis from a conducting plane at $z = 0$ and your result for surface charge density.

a) Form an integral equation for the potential at an observation in the solution region in terms of a volume integral and a surface charge integral on $z = 0$.

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')G(\bar{x},\bar{x}')d^3x' + \frac{1}{4\pi}\oint_S\left[G(\bar{x},\bar{x}')\frac{\delta\Phi}{\delta n'} - \Phi(\bar{x}')\frac{\delta G(\bar{x},\bar{x}')}{\delta n'}\right]da'$$

b) Plug in the free space Green's function.

$$\Phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')\frac{1}{|\bar{x}-\bar{x}'|}d^3x' + \frac{1}{4\pi}\oint_S\left[\frac{1}{|\bar{x}-\bar{x}'|}\frac{\delta\Phi}{\delta n'} - \Phi(\bar{x}')\frac{\delta}{\delta n'}(\frac{1}{|\bar{x}-\bar{x}'|})\right]da'$$

c) Show that when the observation point for the potential is at the position of the charge q that the integral over the surface charge is the same as that contributed by the image charge in the image charge solution method.

$\sigma = D/\varepsilon$ and the integral is that from 2.1 => -q/8πa

d) Show at the image point location (outside the solution region), that the surface integral gives a potential contribution that is equal to and opposite in sign to that from the charge in free space and hence the field is zero.

For $z = -a$ surface integral is same and is cancelled by q giving +q/8πa

e) Bonus: For any arbitrary locations $z < 0$, find an algebraic way to manipulate the integrand to show that the surface charge always makes a contribution that exactly cancels that made directly by charge q and hence the potential is zero everywhere for $z < 0$.

Here one might try an observation position (0,0,b). This gives an integral in which at a point on the screen the denominator of the integrand is b/[(distance to b)(distance to a)³]. This would need to be equal to 1/(a+b) when integrated.