EECS 210   Applied Electromagnetic Theory

Fall 2006   **Office Hours**   **Prof. A. R. Neureuther,**
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Homework #1: Solution

1.1) **Divergence Theorem**: Consider a vector $\vec{A} = x\vec{x} + y\vec{y}$ and a three dimensional rectangular box that goes from (x=0, y=0, z=0) to (x=a, y=b, z=c).

The divergence is 2 and the volume of the box is abc = 2abc.

The dot product of the vector with the local outward normal is a on x = a and of area bc = abc.

The dot product of the vector with the local outward normal is b on y = b and of area ac = abc.

1.2) **Gauss Law Projected on Unit Sphere**: Consider a surface define by

$x^2 + (y - 0.5)^2 + z^2 = 4$ and a charge at (0,0,0). Evaluate Gauss's law over this surface as follows.

a) The integral can be evaluated directly by setting up an integral over two of the three dimensions and using the 3D ellipse formula to eliminate the third variable. It is necessary to find the normal to the ellipse (probably the gradient which is another mess) and compute the dot product with the D field (a third mess) before integrating. So this approach is not recommended.

b) The more tractable approach is that in Figure 1.2 and 1.7 of Jackson.

- Put the point P in Figure 1.7 at the location of the charge and draw a sphere of radius r smaller that would be contained by the ellipse.

- Now approximate the ellipse with a set of small flat areas that intersect and together cover and converge to the entire ellipse as their areas are made smaller.

- Consider one of these small flat areas dA on the ellipse that happens to be a distance R from the charge. The normal to this area is the normal to the ellipse. The dot product of this normal with the radial outward direction from the point charge is cos θ.

- Now consider the projection of dA on to the small sphere of radius r. Since this area is perpendicular to r the dot product of the normal to the sphere and the D field is unity. The area will also scale by r^2/R^2. Combining the scaling due to the normals and the distance gives da = (r^2/R^2)(cos θ)dA.

- One last mapping is that |D| scales by (R^2/r^2) in going from the location of dA to the location of da and by centering the sphere about the charge it is constant at $q/(4\pi r^2)$

- Since the set of dA's on the ellipse produce a corresponding set of da's on the sphere, we can change the summation over the ellipse to the summation over the small sphere using the scaling of both the area and |D| to get an integral over the small sphere.

- Sum over ellipse $|D_{ELLIPSE}|$(cos θ)dA = sum small sphere $\{|D_{ELLIPSE}|(R^2/r^2)\}$ $\{(r^2/R^2)$(cos θ)dA} which is sum small sphere $\{|D_{SPHERE}|\}\{da\}$ = sum small sphere $q/(4\pi r^2)$da =q

1.3) **Integral over a singularity**: Consider a charge at the origin and 4 other charges at arbitrary locations. Take a vanishingly small sphere about the origin of radius ε and integrate both sides of Poisson's equation for the potential over this volume.

      a) Outline an argument as to why the integral over the contributions from the 4 distant charges will be vanishingly small.

- Since the charges are distant the integral can be bounded by taking the maximum potential on the sphere from all four potentials combined and multiplying by the volume of the sphere. As the radius of the sphere ε decrease the integral over the four distant charges goes to zero as $O{\sim}MAX\varepsilon^3$.

      b) Use spherical coordinates and the formulas inside the front cover of Jackson to evaluate the Laplacian operating on the 1/r behavior of the potential as a function of radius and note the dependence on radius.

- Negative gradient of potential $1/(4\pi r) = 1/(4\pi r^2)$ in positive radial direction. Front cover shows divergence is $(2/r)\,[1/(4\pi r^2)] + [(-)2r/(4\pi r^2)] = 0$ THIS WAS A SURPRISE BUT IT IS EXPECTED

      c) Show that this singular behavior can be integrated and gives the desired 4π factor on the right hand side to multiply ρ/ε.

- One approach is to invoke the Divergence theorem. This is straightforward but not all that informative.
- A second approach is to use the a-potential of Jackson. Since we are integrating on a small volume of radius ε which is going to become vanishingly small, the distance a must always be vanishingly small compared to ε. Thus we have nested limits of a disappearing first and then ε disappearing.
- Jackson on page 35 Equation 1.30 carries out this integral involving the Laplacian to get an integral over the charge times a function in which the numerator is proportional to a^2 and the denominator (Assuming r => e) is proportional to r^5.
- Our integration on the small volume only goes in r to ε but we can integrate to infinity is we can argue that the part from ε to infinity vanishes as a and ε go to zero. (Note that Jackson's integral goes to R but he plugs in the result from integrating to infinity.)
- The argument goes as follows; The numerator has a factor $r^2\sin q$ so that the ratio is r^{-3}. This integrates to r^{-2} giving the value for integration from ε to infinity of $\sim(a^2/\varepsilon^2)$. When we let a go to zero first this integration from ε to infinity vanishes.
- {If you can follow these arguments you would do well as a grad student in math.!}