EE C128 / ME C134 – Feedback Control Systems

Lecture – Chapter 4 – Time Response

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September 10, 2013

Lecture abstract

Topics covered in this presentation

- Poles & zeros
- First-order systems
- Second-order systems
- Effect of additional poles
- Effect of zeros
- Effect of nonlinearities
- Laplace transform solution of state equations

Chapter outline

- 4 Time response
  - 4.1 Introduction
  - 4.2 Poles, zeros, and system response
  - 4.3 First-order systems
  - 4.4 Second-order systems: introduction
  - 4.5 The general second-order system
  - 4.6 Underdamped second-order systems
  - 4.7 System response with additional poles
  - 4.8 System response with zeros
  - 4.9 Effects of nonlinearities upon time responses
  - 4.10 Laplace transform solution of state equations
  - 4.11 Time domain solution of state equations

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Definitions, [1, p. 163]

Poles of a TF

- Values of the Laplace transform variable, s, that cause the TF to become infinite
- Any roots of the denominator of the TF that are common to the roots of the numerator

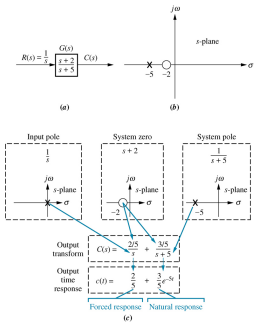

Figure: a. system showing input & output, b. pole-zero plot of the system; c. evolution of a system response

Definitions, [1, p. 163]

Zeros of a TF

- Values of the Laplace transform variable, s, that cause the TF to become zero
- Any roots of the numerator of the TF that are common to the roots of the denominator

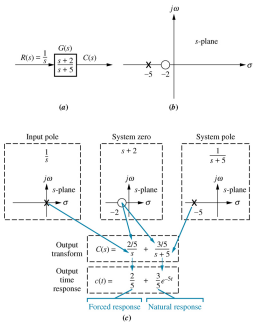

Figure: a. system showing input & output, b. pole-zero plot of the system; c. evolution of a system response

System response characteristics, [1, p. 163]

- *Poles of a TF:* Generate the form of the *natural response*
- *Poles of a input function:* Generate the form of the *forced response*

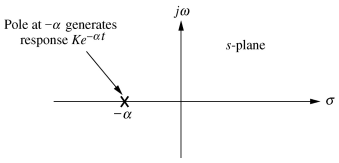

Figure: Effect of a real-axis pole upon transient response

System response characteristics, [1, p. 163]

- *Pole on the real axis:* Generates an *exponential response* of the form $e^{-\alpha t}$, where $-\alpha$ is the pole location on the real axis. The farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero.
- *Zeros and poles:* Generate the *amplitudes* for both the forced and natural responses

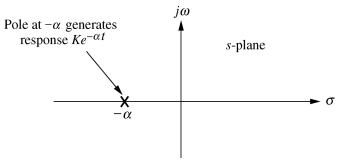

Figure: Effect of a real-axis pole upon transient response

- 4 Time response
  - 4.1 Introduction
  - 4.2 Poles, zeros, and system response
  - 4.3 First-order systems
  - 4.4 Second-order systems: introduction
  - 4.5 The general second-order system
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Intro, [1, p. 166]

- 1^{st}-order system without zeros TF

$$G(s) = \frac{C(s)}{R(s)} = \frac{a}{s+a}$$

- Unit step input TF

$$R(s) = s^{-1}$$

- System response in frequency domain

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$

- System response in time domain

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

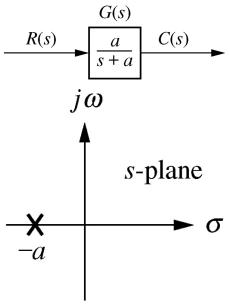

Figure: 1^{st}-order system; pole-plot

Characteristics, [1, p. 166]

- *Time constant,* $\frac{1}{a}$*:* The time for e^{-at} to decay to 37% of its initial value. Alternatively, the time it takes for the step response to rise to 63% of its final value.

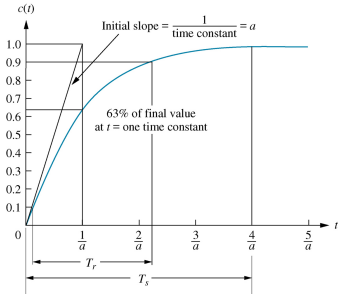

Figure: 1^{st}-order system response to a unit step

Characteristics, [1, p. 166]

- *Exponential frequency,* a*:* The reciprocal of the time constant. The initial rate of change of the exponential at $t = 0$, since the derivative of e^{-at} is $-a$ when $t = 0$. Since the pole of the TF is at $-a$, the farther the pole is from the imaginary axis, the faster the transient response.

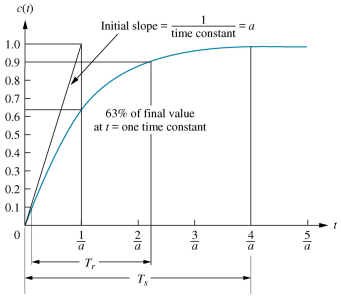

Figure: 1^{st}-order system response to a unit step

Characteristics, [1, p. 166]

- *Rise time, T_r:* The time for the waveform to go from 0.1 to 0.9 of its final value. The difference in time between $c(t) = 0.9$ and $c(t) = 0.1$.

$$T_r = \frac{2.2}{a}$$

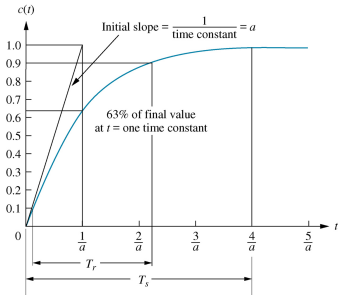

Figure: 1^{st}-order system response to a unit step

Characteristics, [1, p. 166]

- *2% Settling time, T_s:* The time for the response to reach, and stay within, 2% (arbitrary) of its final value. The time when $c(t) = 0.98$.

$$T_s = \frac{4}{a}$$

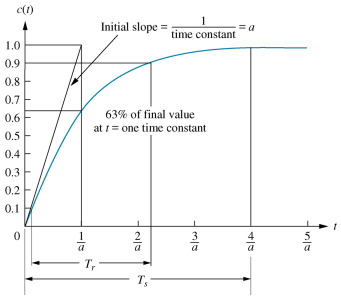

Figure: 1^{st}-order system response to a unit step

- 4 Time response
  - 4.1 Introduction
  - 4.2 Poles, zeros, and system response
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General form, [1, p. 168]

- *2 finite poles:* Complex pole pair determined by the parameters a and b
- *No zeros*

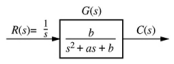


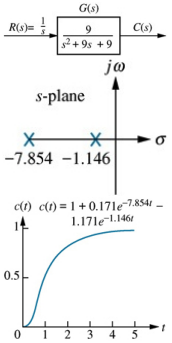
Figure: General 2^{nd}-order system

Overdamped response, [1, p. 169]

- 1 pole at origin from the unit step input
- System poles: 2 real at σ_1, σ_2
- Natural response: Summation of 2 exponentials

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

- Time constants: $-\sigma_1$, $-\sigma_2$

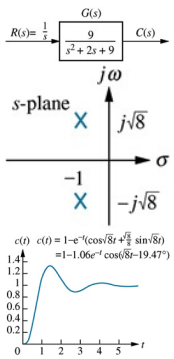


Underdamped response, [1, p. 169]

- 1 pole at origin from the unit step input
- System poles: 2 complex at $\sigma_d \pm j\omega_d$
- Natural response: Damped sinusoid with an exponential envelope

$$c(t) = K_1 e^{-\sigma_d t} cos(\omega_d t - \phi)$$

- Time constant: σ_d
- Frequency (rad/s): ω_d



Underdamped response characteristics, [1, p. 170]

- *Transient response:* Exponentially decaying amplitude generated by the real part of the system pole times a sinusoidal waveform generated by the imaginary part of the system pole.
- *Damped frequency of oscillation, ω_d:* The imaginary part part of the system poles.
- *Steady state response:* Generated by the input pole located at the origin.
- *Underdamped response:* Approaches a steady state value via a transient response that is a damped oscillation.

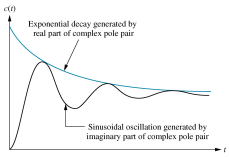


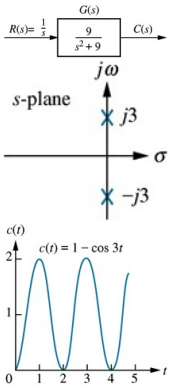
Figure: 2^{nd}-order step response components generated by complex poles

Undamped response, [1, p. 169]

- 1 pole at origin from the unit step input
- System poles: 2 imaginary at $\pm j\omega_1$
- Natural response: Undamped sinusoid

$$c(t) = A\cos(\omega_1 t - \phi)$$

- Frequency: ω_1

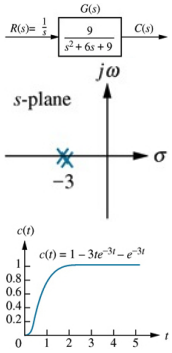


Critically damped response, [1, p. 169]

- 1 pole at origin from the unit step input
- System poles: 2 multiple real
- Natural response: Summation of an exponential and a product of time and an exponential

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$

- Time constant: σ_1
- Note: Fastest response without overshoot



Step response damping cases, [1, p. 172]

- Overdamped
- Underdamped
- Undamped
- Critically damped

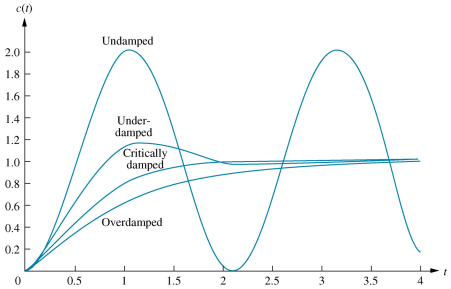


Figure: Step responses for 2^{nd}-order system damping cases

- 4 Time response
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Specification, [1, p. 173]

- *Natural frequency, ω_n*
  - The frequency of oscillation of the system without damping
- *Damping ratio, ζ*

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/s)}} = \frac{1}{2\pi}\frac{\text{Natural period (s)}}{\text{Exponential time constant}}$$

- *General TF*

$$G(s) = \frac{b}{s^2 + as + b} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where

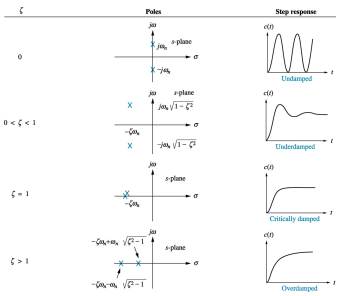
$$a = 2\zeta\omega_n, \quad b = \omega_n^2, \quad \zeta = \frac{a}{2\omega_n}, \quad \omega_n = \sqrt{b}$$

Response as a function of ζ, [1, p. 175]

- *Poles*

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Table: 2^{nd}-order response as a function of damping ratio



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Step response, [1, p. 177]

Transfer function

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

...partial fraction expansion...

$$= \frac{1}{s} + \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

Step response, [1, p. 177]

Time domain via inverse Laplace transform

$$c(t) = 1 - e^{\zeta\omega_n t}\left(\cos\left(\omega_n\sqrt{1-\zeta^2}\right)t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)$$

...trigonometry & exponential relations...

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2} - \phi)$$

where

$$\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

Responses for ζ values, [1, p. 178]

Response versus ζ plotted along a time axis normalized to ω_n

- Lower ζ produce a more oscillatory response
- ω_n does not affect the nature of the response other than scaling it in time

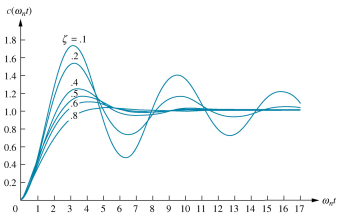

Figure: 2^{nd}-order underdamped responses for damping ratio values

Response specifications, [1, p. 178]

- *Rise time, T_r:* Time required for the waveform to go from 0.1 of the final value to 0.9 of the final value
- *Peak time, T_p:* Time required to reach the first, or maximum, peak

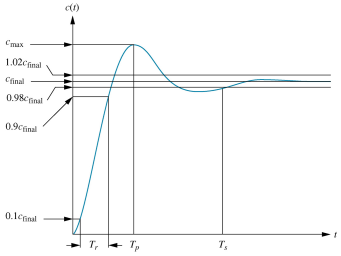

Figure: 2^{nd}-order underdamped response specifications

Response specifications, [1, p. 178]

- *Overshoot, $\%OS$:* The amount that the waveform overshoots the steady state, or final, value at the peak time, expressed as a percentage of the steady state value
- *Settling time, T_s:* Time required for the transient's damped oscillations to reach and stay within $\pm 2\%$ of the steady state value

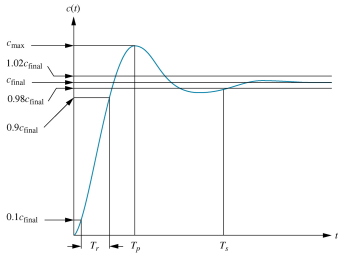

Figure: 2^{nd}-order underdamped response specifications

Evaluation of T_p, [1, p. 179]

T_p is found by differentiating $c(t)$ and finding the zero crossing after $t = 0$, which is simplified by applying a derivative in the frequency domain and assuming zero initial conditions.

$$\mathcal{L}[\dot{c}(t)] = sC(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

...completing the squares in the denominator
...setting the derivative to zero

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Evaluation of $\%OS$, [1, p. 180]

$\%OS$ is found by evaluating

$$\%OS = \frac{c_{\text{max}} - c_{\text{final}}}{c_{\text{final}}} \times 100$$

where

$$c_{\text{max}} = c(T_p), \quad c_{\text{final}} = 1$$

...substitution

$$\%OS = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

ζ given $\%OS$

$$\zeta = \frac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + \ln^2(\frac{\%OS}{100})}}$$

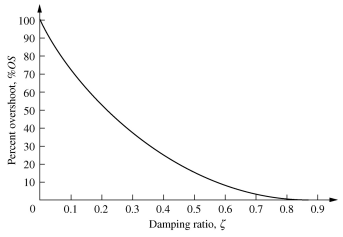

Figure: $\%OS$ vs. ζ

Evaluation of T_s, [1, p. 179]

Find the time for which $c(t)$ reaches and stays within $\pm 2\%$ of the steady state value, c_{final}, i.e., the time it takes for the amplitude of the decaying sinusoid to reach 0.02

$$e^{-\zeta\omega_n t}\frac{1}{\sqrt{1-\zeta^2}} = 0.02$$

This equation is a conservative estimate, since we are assuming that

$$\cos(\omega_n\sqrt{1-\zeta^2}t - \phi) = 1$$

Settling time

$$T_s = \frac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n}$$

Approximated by

$$T_s = \frac{4}{\zeta\omega_n}$$

Evaluation of T_r, [1, p. 181]

A precise analytical relationship between T_r and ζ cannot be found. However, using a computer, T_r can be found

1. Designate $\omega_n t$ as the normalized time variable
2. Select a value for ζ
3. Solve for the values of $\omega_n t$ that yield $c(t) = 0.9$ and $c(t) = 0.1$
4. The normalized rise time $\omega_n T_r$ is the difference between those two values of $\omega_n t$ for that value of ζ

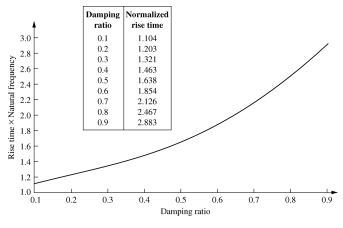

Figure: Normalized T_r vs. ζ for a 2^{nd}-order underdamped response

Location of poles, [1, p. 182]

- *Natural frequency, ω_n:* Radial distance from the origin to the pole
- *Damping ratio, ζ:* Ratio of the magnitude of the real part of the system poles over the natural frequency

$$\cos(\theta) = \frac{-\zeta\omega_n}{\omega_n} = \zeta$$

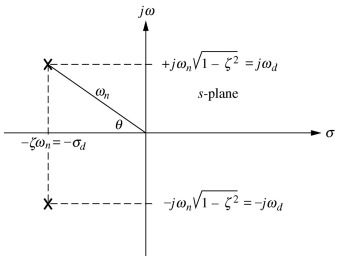

Figure: Pole plot for an underdamped 2^{nd}-order system

Location of poles, [1, p. 182]

- *Damped frequency of oscillation, ω_d:* Imaginary part of the system poles

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

- *Exponential damping frequency, σ_d:* Magnitude of the real part of the system poles

$$\sigma_d = \zeta \omega_n$$

- *Poles*

$$s_{1,2} = -\sigma_d \pm j\omega_d$$

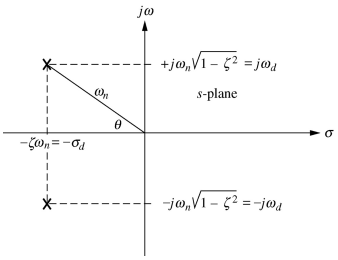

Figure: Pole plot for an underdamped 2^{nd}-order system

Location of poles, [1, p. 183]

- $T_p \propto$ *horizontal lines*

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

- $T_s \propto$ *vertical lines*

$$T_s = \frac{4}{\zeta \omega_n} = \frac{4}{\sigma_d}$$

- $\%OS \propto$ *radial lines*

$$\%OS = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

$$\zeta = \cos(\theta)$$

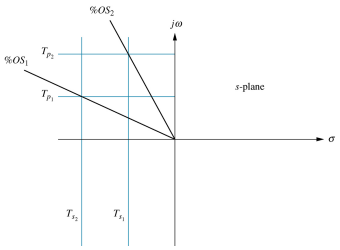

Figure: Lines of constant T_p, T_s, and $\%OS$. Note: $T_{s_2} < T_{s_1}$, $T_{p_2} < T_{p_1}$, $\%OS_1 < \%OS_2$.

Underdamped systems, [1, p. 184]

- $T_p \propto$ *horizontal lines*

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

- $T_s \propto$ *vertical lines*

$$T_s = \frac{4}{\zeta \omega_n} = \frac{4}{\sigma_d}$$

- $\%OS \propto$ *radial lines*

$$\%OS = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

$$\zeta = \cos(\theta)$$

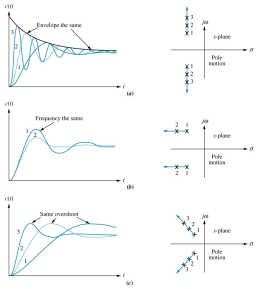

Figure: Step responses of 2^{nd}-order systems as poles move: a. with constant real part, b. with constant imaginary part, c. with constant ζ

- 4 Time response
  - 4.1 Introduction
  - 4.2 Poles, zeros, and system response
  - 4.3 First-order systems
  - 4.4 Second-order systems: introduction
  - 4.5 The general second-order system
  - 4.6 Underdamped second-order systems
  - 4.7 System response with additional poles
  - 4.8 System response with zeros
  - 4.9 Effects of nonlinearities upon time responses
  - 4.10 Laplace transform solution of state equations
  - 4.11 Time domain solution of state equations

Effect on the 2^{nd}-order system, [1, p. 187]

- *Dominant poles:* The two complex poles that are used to approximate a system with more than two poles as a second-order system
- *Conditions:* Three pole system with complex poles and a third pole on the real axis

$$s_{1,2} = \zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}, \quad s_3 = -\alpha_r$$

Effect on the 2^{nd}-order system, [1, p. 187]

- Step response of the system in the frequency domain

$$C(s) = \frac{A}{s} + \frac{B(s+\zeta\omega_n) + C\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{D}{s+\alpha_r}$$

- Step response of the system in the time domain

$$c(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos(\omega_d t) + C\sin(\omega_d t)) + De^{-\alpha_r t}$$

Effect on the 2^{nd}-order system, [1, p. 188]

3 cases for the real pole, α_r

- α_r is not much greater than $\zeta\omega_n$
- $\alpha_r \gg \zeta\omega_n$
  - Assuming exponential decay is negligible after 5 time constants
  - The real pole is $5\times$ farther to the left than the dominant poles
- $\alpha_r = \infty$

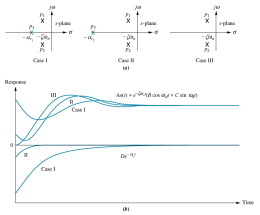

Figure: Component responses of a 3-pole system: a. pole plot, b. component responses: non-dominant pole is near dominant 2^{nd}-order pair, far from the pair, and at ∞

- 4 Time response
  - 4.1 Introduction
  - 4.2 Poles, zeros, and system response
  - 4.3 First-order systems
  - 4.4 Second-order systems: introduction
  - 4.5 The general second-order system
  - 4.6 Underdamped second-order systems
  - 4.7 System response with additional poles
  - 4.8 System response with zeros
  - 4.9 Effects of nonlinearities upon time responses
  - 4.10 Laplace transform solution of state equations
  - 4.11 Time domain solution of state equations

Effect on the 2^{nd}-order system, [1, p. 191]

- *Effects on the system response*
  - Residue, or amplitude
  - Not the nature, e.g., exponential, damped sinusoid, etc.
  - Greater as the zero approaches the dominant poles
- *Conditions:* Real axis zero added to a two-pole system

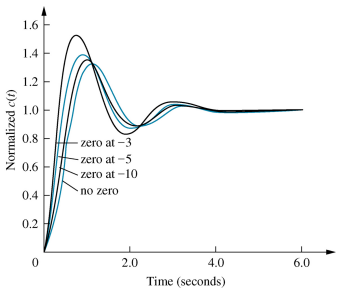

Figure: Effect of adding a zero to a 2-pole system

Effect on the 2^{nd}-order system, [1, p. 191]

Assume a group of poles and a zero far from the poles.
...partial-fraction expansion...

$$\begin{aligned} T(s) &= \frac{s+a}{(s+b)(s+c)} \\ &= \frac{A}{s+b} + \frac{B}{s+c} \\ &= \frac{(-b+a)/(-b+c)}{s+b} + \frac{(-c+a)/(-c+b)}{s+c} \end{aligned}$$

If the zero is far from the poles, then $a \gg b$ and $a \gg c$, and

$$\begin{aligned} T(s) &\approx a\left\{\frac{1/(-b+c)}{s+b} + \frac{1/(-c+b)}{s+c}\right\} \\ &= \frac{a}{(s+b)(s+c)} \end{aligned}$$

Zero looks like a simple gain factor and does not change the relative amplitudes of the components of the response.

Effect on the 2^{nd}-order system, [1, p. 191]

Another view...

- Response of the system, $C(s)$
- System TF, $T(s)$
- Add a zero to the system TF, yielding, $(s+a)T(s)$
- Laplace transform of the response of the system

$$(s+a)C(s) = sC(s) + aC(s)$$

- Response of the system consists of 2 parts
  - The derivative of the original response
  - A scaled version of the original response

Effect on the 2^{nd}-order system, [1, p. 191]

3 cases for a

- a is very large
  - Response $\rightarrow aC(s)$, a scaled version of the original response
- a is not very large
  - Response has additional derivative component producing more overshoot
- a is negative – right-half plane zero
  - Response has additional derivative component with an opposite sign from the scaled response term

Non-minimum-phase system, [1, p. 192]

Non-minimum-phase system:
System that is causal and stable whose inverses are causal and unstable.

- *Characteristics:* If the derivative term, $sC(s)$, is larger than the scaled response, $aC(s)$, the response will initially follow the derivative in the opposite direction from the scaled response.

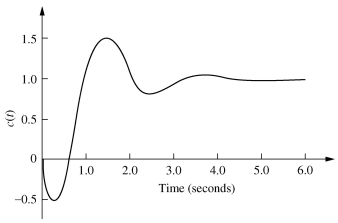

Figure: Step response of a non-minimum-phase system

- 4 Time response
  - 4.1 Introduction
  - 4.2 Poles, zeros, and system response
  - 4.3 First-order systems
  - 4.4 Second-order systems: introduction
  - 4.5 The general second-order system
  - 4.6 Underdamped second-order systems
  - 4.7 System response with additional poles
  - 4.8 System response with zeros
  - 4.9 Effects of nonlinearities upon time responses
  - 4.10 Laplace transform solution of state equations
  - 4.11 Time domain solution of state equations

Saturation, [1, p. 196]

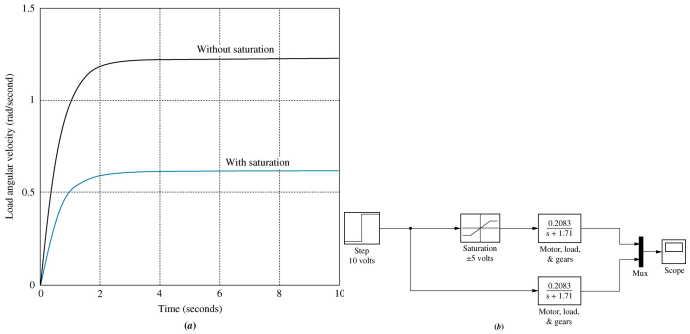

Figure: a. effect of amplifier saturation on load angular velocity response, b. Simulink block diagram

Dead zone, [1, p. 197]

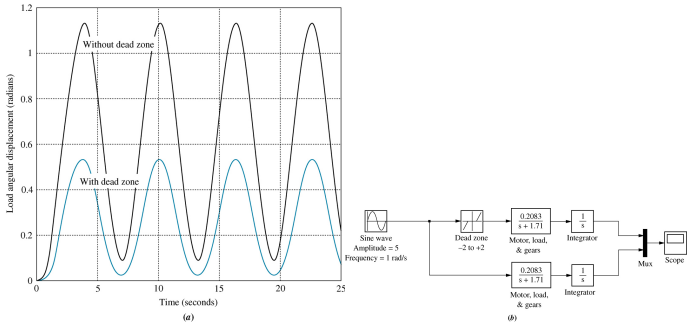

Figure: a. effect of dead zone on load angular displacement response, b. Simulink block diagram

Backlash, [1, p. 198]

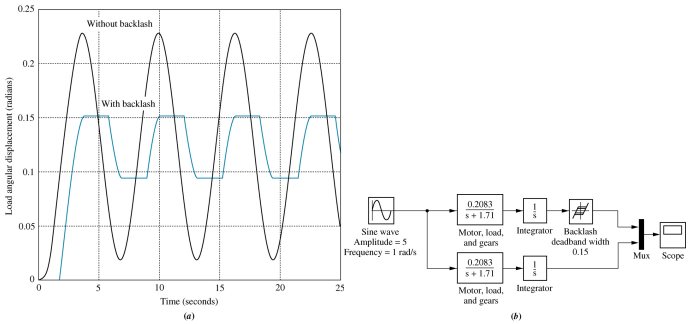

Figure: a. effect of backlash on load angular displacement response, b. Simulink block diagram

- 4 Time response
  - 4.1 Introduction
  - 4.2 Poles, zeros, and system response
  - 4.3 First-order systems
  - 4.4 Second-order systems: introduction
  - 4.5 The general second-order system
  - 4.6 Underdamped second-order systems
  - 4.7 System response with additional poles
  - 4.8 System response with zeros
  - 4.9 Effects of nonlinearities upon time responses
  - 4.10 Laplace transform solution of state equations
  - 4.11 Time domain solution of state equations

Laplace transform solution of state equations, [1, p. 199]

State equation

$$\dot{x} = Ax + Bu$$

Output equation

$$y = Cx + Du$$

Laplace transform of the state equation

$$zX(s) - x(0) = AX(s) + BU(s)$$

...combining all the $X(s)$ terms

$$(sI - A)X(s) = x(0) + BU(s)$$

...solving for $X(s)$

$$\begin{aligned} X(s) &= (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) \\ &= \frac{\operatorname{adj}(sI - A)}{\det(sI - A)}\left[x(0) + BU(s)\right] \end{aligned}$$

Laplace transform solution of state equations, [1, p. 199]

State equation

$$\dot{x} = Ax + Bu$$

Output equation

$$y = Cx + Du$$

Laplace transform of the state equation

$$Y(s) = CX(s) + DU(s)$$

Eigenvalues & TF poles, [1, p. 200]

Eigenvalues of the system matrix A are found by evaluating

$$\det(sI - A) = 0$$

The eigenvalues are equal to the poles of the system TF.
...for simplicity, let the output, $Y(s)$, and the input, $U(s)$, be scalar quantities, and further, to conform to the definition of a TF, let $x(0) = 0$

$$\begin{aligned} \frac{Y(s)}{U(s)} &= C\left[\frac{\operatorname{adj}(sI-A)}{\det(sI-A)}\right]B + D \\ &= \frac{C\operatorname{adj}(sI - A)B + D\det(sI - A)}{\det(sI - A)} \end{aligned}$$

The roots of the denominator are the poles of the system.

- 4 Time response
  - 4.1 Introduction
  - 4.2 Poles, zeros, and system response
  - 4.3 First-order systems
  - 4.4 Second-order systems: introduction
  - 4.5 The general second-order system
  - 4.6 Underdamped second-order systems
  - 4.7 System response with additional poles
  - 4.8 System response with zeros
  - 4.9 Effects of nonlinearities upon time responses
  - 4.10 Laplace transform solution of state equations
  - 4.11 Time domain solution of state equations

Time domain solution of state equations, [1, p. 203]

Linear time-invariant (LTI) system

- *Time domain solution*

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- *Zero-input response*

$$e^{At}x(0)$$

- *Zero-state response / convolution integral*

$$\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- *State-transition matrix*

$$e^{At}$$

Time domain solution of state equations, [1, p. 203]

Linear time-invariant (LTI) system

- *Frequency domain unforced response*

$$\mathcal{L}[x(t)] = \mathcal{L}[e^{At}x(0)] = (sI - A)^{-1}x(0)$$

- *Laplace transform of the state-transition matrix*

$$e^{At}x(0)$$

- *Characteristic equation*

$$\det(sI - A) = 0$$

- *??? equation*

$$\mathcal{L}^{-1}[(sI - A)^{-1}] = \mathcal{L}^{-1}\left[\frac{\operatorname{adj}(sI-A)}{\det(sI-A)}\right] = \Phi(t)$$

Bibliography

Norman S. Nise. *Control Systems Engineering*, 2011.