Lab 5c: LQR Controller Design for Inverted Pendulum

I. Purpose

The objective of this lab is to design a full-state feedback controller using the Linear Quadratic Regulator (LQR) design technique.

II. Theory

Pole placement for controller design relies on specification of the desired closed-loop poles of the system. This is usually difficult to specify, especially for systems with a large number of states. Furthermore, with pole placement design there is no consideration given to the "amount" of actuation (called actuation or control effort) that gets used during closed-loop operation.

Good regulation of the system can usually be achieved by using high amount of actuation (i.e. higher K_p, and thus greater actuation effort, in a P-controller gives faster rise time). But in reality, we are often limited by power and energy constraints. Ideally, we would like to achieve good system performance while at the same time minimizing the amount of actuation used in achieving the desired performance. One way of expressing this mathematically is through an objective function of the form:

$$J = \int_0^\infty (x^T Q x + u^T R u)\, dt$$

The LQR design problem is to design a state-feedback controller K (i.e. for $u = -Kx$) such that the objective J is minimized. The cost functional J above consists of two terms, the first of which you can think of as being the cost of regulating the state x (regulatory term) and the second being the cost of actuation (actuation term). Both of these terms has an associated free matrix, Q and R respectively. The regulatory term will "penalize" you for any deviations from your desired state, while the actuation term will "penalize" you for any actuation effort.

For simplicity we assume that matrices Q and R are diagonal. Thus, the objective J reduces to:

$$J = \int_0^\infty ({q_1 x_1}^2 + \cdots + {q_n x_n}^2 + {r_1 u_1}^2 + \cdots + {r_m u_m}^2) dt$$

Here, the scalars $q_1, \ldots, q_n, r_1, \ldots, r_m$ can be looked upon as relative weights between different performance terms in the objective J. The key design problem in LQR is to translate performance specifications in terms of the rise time, overshoot, bandwidth, etc. into relative weights of the above form. There is no straightforward way of doing this and it is usually done through an iterative process either in simulations or on an experimental setup.

Once the matrices Q and R are completely specified, the controller gain K is found by solving the Riccati equation, which we will do in MATLAB.

III. Pre-lab

The model for the inverted pendulum system remains the same as in Lab 5a and 5b. We have a four-state model with states $x, \dot{x}, \theta, \dot{\theta}$ and one input V. **What will the dimensions of Q and R be?** The prelab mainly consists of translating the performance specifications stated into matrices Q and R. We assume the LQR matrices are diagonal.

- The objective of the controller could *hypothetically* be stated as follows:

  Given that the cart and the pendulum are 30 cm and 0.05 radians (≈ 2.5 deg) displaced from their desired positions at time $t = 0$, the objective is to get the system to the desired state as soon as possible but without using, say, more than 8 volts of the input at any point in time. For now, however, we will **ignore the constraint on input**.

- For our problem, we set scalars q_2 and q_4 to 0 as we desire no restriction on how $\dot{x}$ and $\dot{\theta}$ vary with time.

  Now, in order to use scalars q_1, q_3 and r as **relative** weights, we will normalize them based on their initial conditions. Thus, the modified weights turn out to be:

$$\bar{q}_1 = \frac{q_1}{0.3^2}, \qquad \bar{q}_3 = \frac{q_3}{0.05^2}, \qquad \bar{r} = \frac{r}{8^2}$$

  The weights have been normalized with square terms because our objective function J is a quadratic function of x and u. (So the matrix Q will use $\bar{q}_1$ and $\bar{q}_3$, and $R = \bar{r}$)

(a) For nominal weights $q_1 = 1$, $q_3 = 1$, and $r = 1$ (giving equal weight to each term of the objective function), come up with the **gain matrix K** which minimizes the objective function and its associated **CL pole locations**. You may use the `lqr` command in MATLAB to do this. Simulate the closed-loop system (with or without estimator) and plot y and u for initial conditions of 30 cm and 0.05 radians.

(b) Individually vary the weights from their nominal values and study the influence of the weights on how the system outputs and control effort varies with time. The weights are relative, so you may assume $q_1 = 1$ in all cases and vary only the other two. For each case:

   1) report K and the CL pole locations
   2) plot y and u,
   2) report the maximum deviations in x and θ as well as u_{max}, and
   3) state the observed influences in words.

   There will be 5 sets of graphs in total (nominal, $q_3 \ll 1$, $q_3 \gg 1$, $r \ll 1$, and $r \gg 1$).

IV. Lab

Implement the controller you designed for the nominal weights and observe the output response of the system. Be consistent with regards to the observer – if you used the observer in the pre-lab, use it here, but if you didn't, then don't use it here.

Now, implement the controller designed for a higher value of q_3 and then another controller designed for a higher value of r.

In each case note the variation of the position of the cart and the pendulum with time and the control input and also observe the influence of the weights visually on the setup, i.e. plot y and u and make qualitative observations. Make sure that the differences are noticeable on your plots.

Show your GSI your running controller in MATLAB and these plots (with your qualitative observations) for check off (5 pt). No lab writeup is required.

V. Revision History

Semester and Revision	Author(s)	Comments
Fall 2010 Rev. 1.3	Wenjie Chen and Jansen Sheng	Modified slightly about the lab requirement (DEMO instead of lab writeup).
Winter 2009 Rev. 1.2	Justin Hsia	Corrections and additions following Fall 2009 run-through.
Winter 2008 Rev. 1.1	Justin Hsia	Corrections and additions following Fall 2008 run-through and feedback.
Fall 2008 Rev. 1.0	Justin Hsia	Converted lab to Word
Fall 2008 Rev. 1.0	Pranav Shah	Initial lab write-up