Lab 5a: Pole Placement for the Inverted Pendulum

I. Purpose

The objective of this lab is to achieve simultaneous control of both the angular position of the pendulum and horizontal position of the cart on the track using full-state feedback. We will be considering small angle perturbations and sine wave reference tracking.

II. Theory

The setup consists of a pendulum attached to a movable cart as shown in figure below.

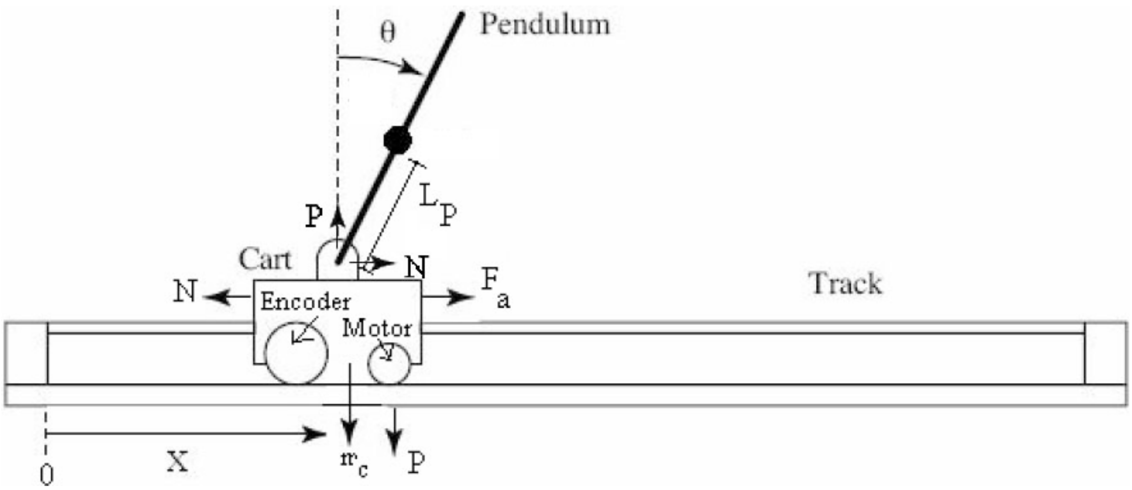

Figure 1: Cart-inverted pendulum setup with free body diagram

We ignore friction and assume that the entire mass of the pendulum is concentrated at its center of mass, which is half-way up the length of the pendulum ($L_p = L/2$). N and P are the horizontal and vertical components, respectively, of the reaction force between the cart and the pendulum.

Here are the values of the physical system:

Symbol	Description	Value
	Resolution of cart position encoder	439.6 counts/cm
	Resolution of angle encoder	666.7 counts/rad
M	Mass of cart and motor	0.94 kg
m	Mass of pendulum	0.230 kg
L_p	Pendulum distance from pivot to center of mass	0.3302 m
I_c	Moment of inertia of pendulum about its center	$mL_p^2/3$
I_e	Moment of inertia of pendulum about its end	$4*mL_p^2/3$
K_t	Motor torque constant	0.00767 N*m/A
K_m	Motor back emf constant	0.00767 V*s/rad
K_g	Motor gearbox ratio	3.71
R_m	Motor winding resistance	2.6 Ω
r	Radius of motor gear	0.00635 m
J_m	Motor moment of inertia	3.9×10^{-7} kg*m^2

III. Pre-lab

a) Derive the equations of motion of the inverted pendulum-cart system. One way of doing this is by considering the free-body diagrams of the cart and the pendulum separately and writing their equations of motion.

(**Hint:** While deriving use the small-signals approximation $\boldsymbol{\sin(\theta) \approx \theta}$ and $\boldsymbol{\cos(\theta) \approx 1}$. This will simplify the math a little.)

The equations should turn out to be the following:

$$(M+m)\ddot{x} + mL_P\ddot{\theta} = F_a \quad (1)$$

$$mL_P\ddot{x} + \frac{4mL_P{}^2}{3}\ddot{\theta} - mgL_P\theta = 0 \quad (2)$$

Here, $\boldsymbol{F_a}$ equals the force exerted on the cart by the attached motor.

b) Use the motor dynamics derived in the Lab 2 (in the form $\boldsymbol{F_a} = f(\boldsymbol{V}, x, \dot{x})$) and substitute this into the cart-pendulum dynamics from part a to obtain the complete system dynamics.

The outputs of our interest are the position of the cart ($\boldsymbol{x}$) and the pendulum angle ($\boldsymbol{\theta}$) and the available control input is the voltage applied to the motor ($\boldsymbol{V}$). Thus, our system is a 1-input, 2-output system (SIMO).

Substitute the model parameters and obtain the state-space model for the complete system. You may use $\boldsymbol{X} = [\boldsymbol{x} \quad \dot{\boldsymbol{x}} \quad \boldsymbol{\theta} \quad \dot{\boldsymbol{\theta}}]^{\boldsymbol{T}}$ as your state vector. Make sure you do this derivation *symbolically*. Once you have the expressions for your state space representation, be sure to use a MATLAB script file to plug in your values. We actually have two different sets of pendulums in the lab, and this will save you a lot of headaches down the road if you need to alter your parameters.

To help you verify your solution, you may check your numerical answers against what is shown below. Derived formula errors tend to propagate across entire rows.

$$A = \begin{bmatrix} * & * & * & * \\ * & -6.8312 & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \qquad B = \begin{bmatrix} * \\ * \\ * \\ -3.4625 \end{bmatrix} \qquad C = ? \qquad D = ?$$

c) Determine the eigenvalues of the state matrix A and the poles of the state-space representation. Is the open-loop system internally stable? BIBO stable? Also, check if the system is controllable and observable. (For this part you may use the MATLAB commands `eig`, `ctrb`, `obsv`, and `rank`)

d) Simulate the output response of the system for a step input in the applied voltage. What would you expect to happen to x and θ in the physical system (assume infinite track length)? What are the discrepancies between the simulation and the physical system? Why might these discrepancies be there?

e) We will use a state-feedback controller to achieve the desired performance specifications. For the purpose of design, we assume that all the state variables are available for measurement and can use them for feedback (i.e. the entire state vector X is known).

The full-state feedback controller is $\boldsymbol{u} = -\boldsymbol{KX}$. The gain matrix K is chosen such that the closed-loop eigenvalues lie at some desired values. These desired values of the eigenvalues are found based on the performance specifications desired to be achieved.

We would like our closed-loop eigenvalues to lie at $-\mathbf{1.7} \pm \mathbf{9}\boldsymbol{j}$ **and** $-\mathbf{2} \pm \mathbf{1.6}\boldsymbol{j}$. Using MATLAB, find the feedback gain matrix K that will help us achieve this. You may use the `acker` or `place` commands for pole placement.

f) Now your system dynamics are $\dot{\boldsymbol{X}} = \boldsymbol{AX} + \boldsymbol{Bu},\ \boldsymbol{u} = \boldsymbol{K}(\boldsymbol{r} - \boldsymbol{X})$, where $\boldsymbol{r}$ is your reference input. Using the calculated gain matrix $\boldsymbol{K}$ and MATLAB, calculate the **closed loop transfer function** from the first component in the reference $\boldsymbol{r}$ (position reference input) to the output $\boldsymbol{x}$ (only the position of the cart). You may use the `ss` and `tf` commands. Plot the Bode plot of this transfer function. You should notice that the DC gain for this transfer function is unity.

IV. Lab

Based on which station you use, you will either have a long or short pendulum. The different pendulum physical properties are listed below:

Symbol	Description	Value
$\boldsymbol{m_l}$	Mass of pendulum (long)	0.230 kg
$\boldsymbol{L_{p,l}}$	Pendulum distance from pivot to center of mass (long)	0.3302 m
$\boldsymbol{m_s}$	Mass of the pendulum (short)	0.127 kg
$\boldsymbol{L_{p,s}}$	Pendulum distance from pivot to center of mass (short)	0.1778 m

a) Implement the designed state-feedback controller on the actual setup (with **saturation limit** 8V and -8V for the voltage input). However, since we have measurements only for the linear and angular position of the cart ($\boldsymbol{x}$) and pendulum ($\boldsymbol{\theta}$) and not their velocities, $\dot{\boldsymbol{x}}$ and $\dot{\boldsymbol{\theta}}$, we can numerically differentiate $\boldsymbol{x}$ and $\boldsymbol{\theta}$ to <u>approximate</u> $\dot{\boldsymbol{x}}$ and $\dot{\boldsymbol{\theta}}$, respectively. This helps obtain the entire state vector $\boldsymbol{X}$ for feedback. Don't forget to include your QuaRC diagram and subsystem (if applicable) in your lab report.

> **Note 1:** The direction we have defined $\boldsymbol{\theta}$ to increase and the direction the angular encoder on the hardware increases are *opposite* of each other. Therefore, make sure your angular unit conversion is **negative**.

> **Note 2:** As always, watch your units! Everything in this system should match the units that you used in your state-space derivation.

b) Run the controller on the hardware (with $\boldsymbol{r}$ set to 0) and make sure it balances. You will need to hold the pendulum vertical before starting the program. Give the pendulum small perturbations manually and check the response.

c) Now we will introduce a sine wave reference position input into the system and analyze the results. Your reference would be like $\boldsymbol{r} = [\boldsymbol{M}\boldsymbol{sin}(\boldsymbol{\omega t})\ \ \mathbf{0}\ \ \mathbf{0}\ \ \mathbf{0}]'$. Start with a reasonable amplitude

($\boldsymbol{M}$ =**0.10 m**), use frequencies ($\boldsymbol{\omega}$) of 1, 5, and 10 rad/sec. Make sure to start at the center of the track! You will also need to hold the pendulum vertical before starting the program. Check your response.

d) Be sure to **save all your data** (especially reference input and position output) from the workspace for your post lab. **Demonstrate to your GSI** your working stage for check off.

V. Post-lab Section

For Part b) in the lab section, plot the variation of the cart and pendulum position with time for small perturbations (manually-induced) about the equilibrium value. Comment on the controller's general performance. **Why does the hardware continue to oscillate about the equilibrium point?**

For Part c) in the lab section, calculate the gain and phase for each of the frequencies in your frequency response (ignoring the offset from the hardware response). Locate these frequencies on the Bode plot from Part f) of your prelab and compare the results. **Do your values match for each frequency? If not, explain possible causes for the difference.**

VI. Revision History

Semester and Revision	Author(s)	Comments
Fall 2010 Rev 1.5	Wenjie Chen and Jansen Sheng	Added a post lab section and modified other sections
Winter 2009 Rev. 1.4	Justin Hsia	Corrections and additions after Fall 2009 run-through.
Fall 2009 Rev. 1.3	Justin Hsia	Updated table of constants for new equations including moment of inertia.
Winter 2008 Rev. 1.2	Justin Hsia	Corrections and additions after Fall 2008 run-through and feedback.
Fall 2008 Rev. 1.1	Justin Hsia	Converted lab to Word, fixed free body diagram and other errors.
Fall 2008 Rev. 1.0	Pranav Shah	Initial lab write-up.