Lecture #18 example (Oct. 28, 2010) (from Tomlin Lecture #8, Fall 2008)

Similarity transforms App E.13: $\mathbf{x} = P\bar{\mathbf{x}}$.
Inverted pendulum

$$ml^2\ddot{\theta} - mgl\sin\theta = T \tag{1}$$

Linearized transfer function:

$$\frac{\Theta(s)}{U(s)} = \frac{1}{s^2 - g/l} = \frac{1}{(s+\Omega)(s-\Omega)} = \tag{2}$$

In state space form with compensator $D(s) = \frac{U(s)}{U_1(s)} = \frac{s-\hat{\Omega}}{s}$:

$$\dot{u}(t) = \dot{u}_1(t) - \hat{\Omega}u_1(t) \tag{3}$$

By integration of eqn(3) we get

$$u(t) = u_1(t) - \hat{\Omega}\int_0^t u_1(\tau)d\tau \tag{4}$$

Introduce state variable $x_3 = \hat{\Omega}\int u_1$, then $\dot{x_3} = \hat{\Omega}u_1(t)$ and $u = u_1 - x_3$. Thus state space representation with compensation is:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \Omega^2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \hat{\Omega} \end{bmatrix} u_1(t) \tag{5}$$

Eigenvalues $|\lambda I - A| = 0$:

$$\begin{vmatrix} \lambda & -1 & 0 \\ -\Omega^2 & \lambda & 1 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda(\lambda^2 - \Omega^2) = 0 \tag{6}$$

Thus eigenvalues are $\lambda_0 = 0$, $\lambda_1 = \Omega$, and $\lambda_2 = -\Omega$. and eigenvectors $(\lambda I - A)\mathbf{e_i} = \mathbf{0}$:

$$\mathbf{e_0} = \begin{bmatrix} 1 \\ 0 \\ \Omega^2 \end{bmatrix} \quad \mathbf{e_1} = \begin{bmatrix} 1 \\ \Omega \\ 0 \end{bmatrix} \quad \mathbf{e_2} = \begin{bmatrix} 1 \\ -\Omega \\ 0 \end{bmatrix} \tag{7}$$

To diagonalize A, let $\bar{A} = P^{-1}AP$, where

$$P = [\mathbf{e_1}\ \mathbf{e_2}\ \mathbf{e_0}] = \begin{bmatrix} 1 & 1 & 1 \\ \Omega & -\Omega & 0 \\ 0 & 0 & \Omega^2 \end{bmatrix} \quad \text{and} \quad \mathrm{P}^{-1} = \frac{1}{2\Omega^2} \begin{bmatrix} \Omega^2 & \Omega & -1 \\ \Omega^2 & -\Omega & -1 \\ 0 & 0 & 2 \end{bmatrix} \tag{8}$$

Thus

$$\bar{A} = P^{-1}AP = \frac{1}{2\Omega^2} \begin{bmatrix} \Omega^2 & \Omega & -1 \\ \Omega^2 & -\Omega & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \Omega^2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ \Omega & -\Omega & 0 \\ 0 & 0 & \Omega^2 \end{bmatrix} = \begin{bmatrix} \Omega & 0 & 0 \\ 0 & -\Omega & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{9}$$

Output

$$\mathbf{y} = \mathbf{Cx} = \mathbf{CP}\bar{\mathbf{x}} = [\mathbf{1}\ \mathbf{0}\ \mathbf{0}] \begin{bmatrix} 1 & 1 & 1 \\ \Omega & -\Omega & 0 \\ 0 & 0 & \Omega^2 \end{bmatrix} \bar{\mathbf{x}} = [\mathbf{1}\ \mathbf{1}\ \mathbf{1}]\bar{\mathbf{x}} \tag{10}$$

$$\tag{11}$$

A variation of the modal canonical form with $\mathbf{y} = \mathbf{Cx}$ is shown here: