Lecture #16 with fixed output equation (Oct. 21, 2010)

Consider SISO LTI system with input $u(t)$ and output $y(t)$ with transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \tag{1}$$

Introduce intermediate function $X(s)$ with

$$Y(s) = (b_0 s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4) X(s) \tag{2}$$

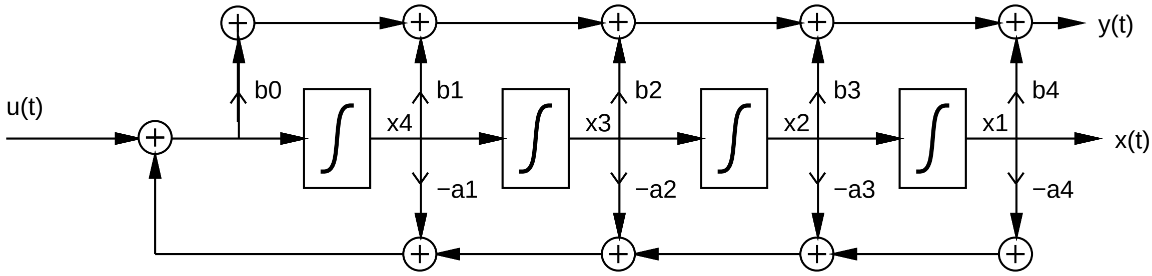
and

$$X(s) = \frac{U(s)}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \ . \tag{3}$$

Using the inverse Laplace transform on eqn. 3 we have

$$\frac{d^4 x}{dt^4} = u - a_1 \frac{d^3 x}{dt^3} - a_2 \frac{d^2 x}{dt^2} - a_3 \frac{dx}{dt} - a_4 x \tag{4}$$

This differential equation for $x(t)$ can be solved using a chain of integrators with feedback as shown here:



where $x = x_1$, $\dot{x_1} = x_2$ $\dot{x_2} = x_3$ $\dot{x_3} = x_4$ and $\dot{x_4} = \frac{d^4 x}{dt^4}$.
Referring to the block diagram, it is easy to write the state equation:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{5}$$

Output Equation

Considering that by the inverse Laplace transform of eqn.(2), the output $y(t)$ is just a linear combination of the derivatives of $x(t)$, we get:

$$y(t) = b_0 \frac{d^4 x}{dt^4} + b_1 \frac{d^3 x}{dt^3} + b_2 \frac{d^2 x}{dt^2} + b_3 \frac{dx}{dt} + b_4 x \ . \tag{6}$$

The output $y(t)$ is shown in the block diagram.

In terms of the state variable assignment we have $y(t) = b_0 \dot{x_4} + b_1 x_4 + b_2 x_3 + b_3 x_2 + b_4 x_1$. However, the output equation $y = C\mathbf{x} + Du$ needs to be in terms of the states, thus we need to calculate $\dot{x_4}$ by using eqn.(4). Hence,

$$y(t) = b_0(u - a_1 x_4 - a_2 x_3 - a_3 x_2 - a_4 x_1) + b_1 x_4 + b_2 x_3 + b_3 x_2 + b_4 x_1 \ . \tag{7}$$

By combining terms, we get

$$y(t) = b_0 u + (b_1 - b_0 a_1) x_4 + (b_2 - b_0 a_2) x_3 + (b_3 - b_0 a_3 x_2) + (b_4 - b_0 a_4) x_1 \tag{8}$$

then the corrected output equation for controllable canonical form is:

$$y = C\mathbf{x} + Du = \begin{bmatrix} b_4 - b_0 a_4 & b_3 - b_0 a_3 & b_2 - b_0 a_2 & b_1 - b_0 a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + b_0 u(t) \tag{9}$$