EE123
Digital Signal Processing

Lecture 25

Based on lecture notes by Prof. Murat Arcak

Announcements

- Miki cannot be here today
- Lab bash during discussion section
- Lab 2 part 2 extended to 11:59pm today

Rational system response

In real life, we get linear difference equations:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Example: $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

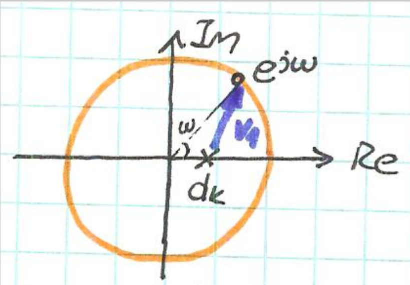
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = |\frac{b_0}{a_0}| \cdot \frac{\prod_{k=0}^{M}|1-c_k e^{-j\omega}|}{\prod_{k=0}^{N}|1-d_k e^{-j\omega}|}$$

Consider one of the poles:

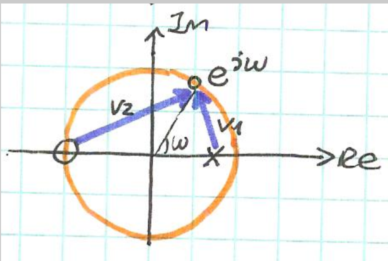
$$|1-d_k e^{-j\omega}| = |e^{+j\omega} - d_k| \;\; = |v_1|$$

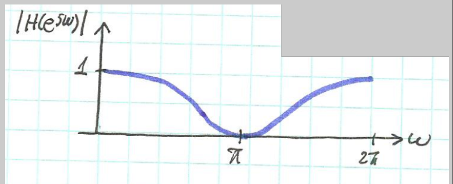


Magnitude Response Example

Example: $$H(z) = 0.05\frac{1+z^{-1}}{1-0.9z^{-1}}$$

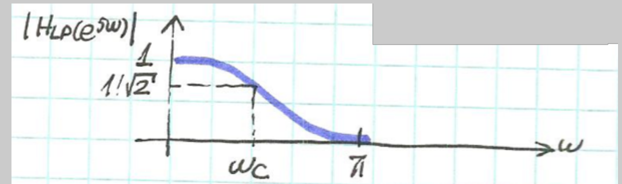
$$|H(z)| = 0.05\frac{|v_2|}{|v_1|}$$





Simple low pass filter

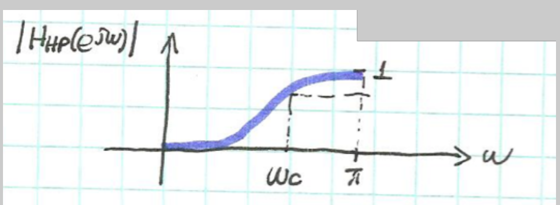
$$H_{LP}(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}} \qquad |\alpha| < 1$$



w_c is the 3dB cutoff frequency $$\alpha = \frac{1-\sin(\omega_c)}{\cos(\omega_c)}$$

Simple high pass filter

$$H_{HP}(z) = \frac{1+\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}} \qquad |\alpha| < 1$$



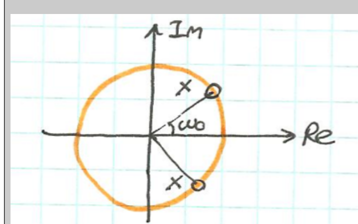
w_c is the 3dB cutoff frequency $$\alpha = \frac{1-\sin(\omega_c)}{\cos(\omega_c)}$$

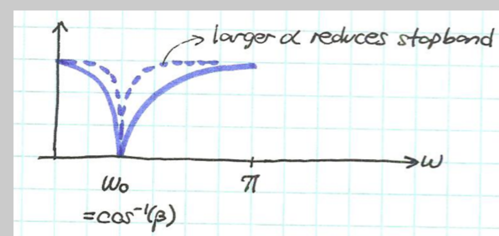
same as low pass

Simple band-stop (Notch) filter

$$H_{BS}(z) = \frac{1+\alpha}{2}\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \qquad \begin{array}{l}|\alpha| < 1\\ |\beta| < 1\end{array}$$

Note: $$1-2\beta z^{-1}+z^{-2} = (1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})$$
$$\cos(\omega_0) = \beta$$



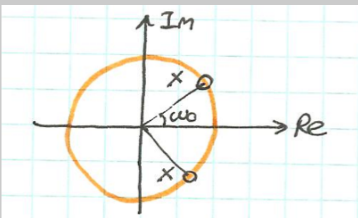


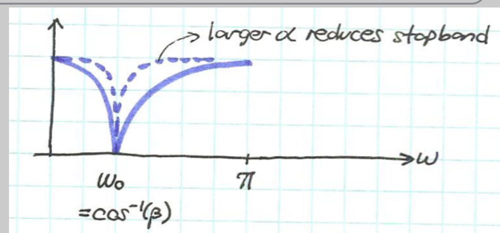
Simple band-stop (Notch) filter

$$H_{BS}(z) = \frac{1+\alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}} \qquad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

Note: As $\alpha \to 1$ poles approach zeros

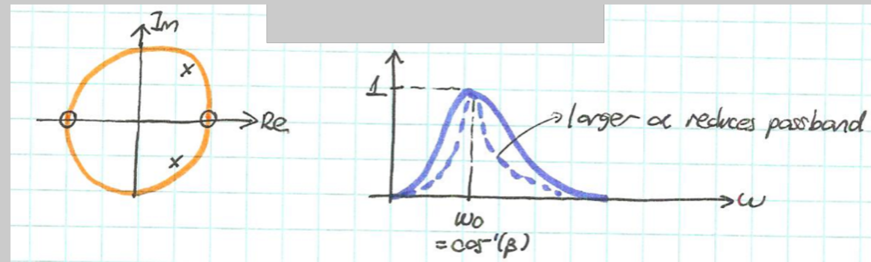
$$H_{BS}(\mp 1) = \frac{1+\alpha}{2} \frac{2 \pm 2\beta}{(1+\alpha)(1 \pm \beta)} = 1$$





Simple band-pass filter

$$H_{BP}(z) = \frac{1-\alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}} \qquad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$



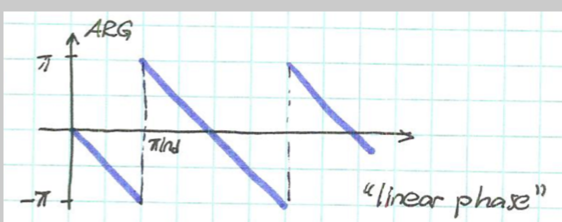
Phase response

Example: $H(e^{j\omega}) = e^{j\omega n_d} \quad \leftrightarrow \quad h[n] = \delta[n - n_d]$

$$|H(e^{j\omega})| = 1$$

$$\arg[H(e^{j\omega})] = -\omega n_d$$

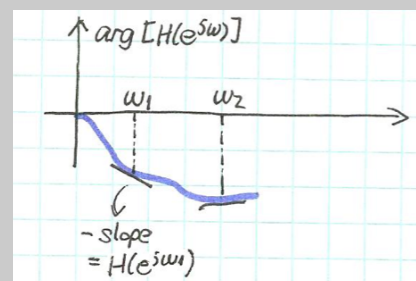
ARG is the wrapped phase
arg is the unwrapped phase



Group delay

To characterize general phase response, look at the group delay:

$$\mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

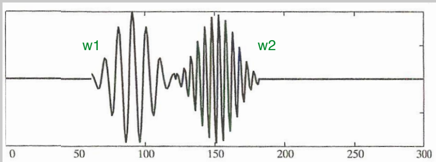


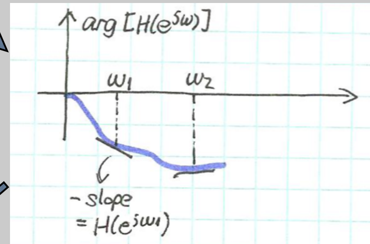
For linear phase system, the group delay is n_d

Group delay

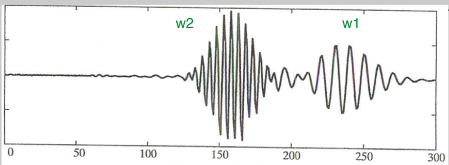
$$\mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

Input





Output



For narrowband signals, phase response looks like a linear phase

Group delay math

$$H(z) = \frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

arg of products is sum of args

$$\arg[H(e^{j\omega})] = -\sum_{k=1}^{N}\arg[1-d_k e^{-j\omega}] + \sum_{k=1}^{M}\arg[1-c_k e^{-j\omega}]$$

$$\mathrm{grd}[H(e^{j\omega})] = -\sum_{k=1}^{N}\mathrm{grd}[1-d_k e^{-j\omega}] + \sum_{k=1}^{M}\mathrm{grd}[1-c_k e^{-j\omega}]$$

Group delay math

$$\mathrm{grd}[H(e^{j\omega})] = -\sum_{k=1}^{N}\mathrm{grd}[1-d_k e^{-j\omega}] + \sum_{k=1}^{M}\mathrm{grd}[1-c_k e^{-j\omega}]$$

Look at each factor:

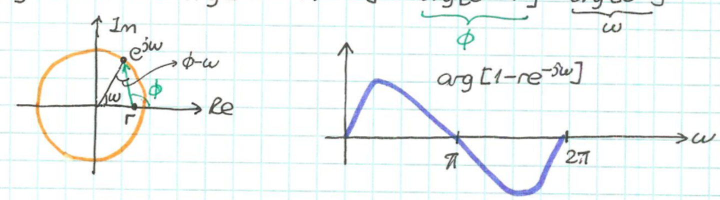
$$\arg[1-\underbrace{re^{j\theta}}_{c_k \text{ or } d_k}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right)$$

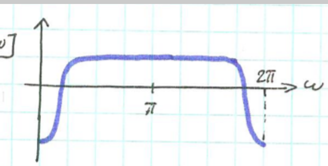
$$\mathrm{grd}[1-re^{j\theta}e^{-j\omega}] = \frac{r^2-r\cos(\omega-\theta)}{|1-re^{j\theta}e^{-j\omega}|^2}$$

Look at a zero lying on the real axis

Geometric Interpretation (for $\theta=0$)

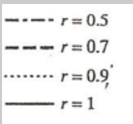
$$\arg[1-re^{-j\omega}] = \arg[(e^{j\omega}-r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega}-r]}_{\phi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



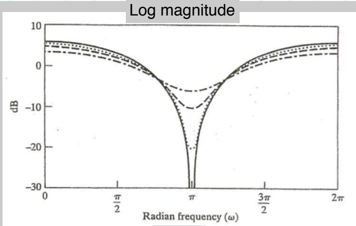


$\theta \neq 0 \Rightarrow$ shift to the right by θ

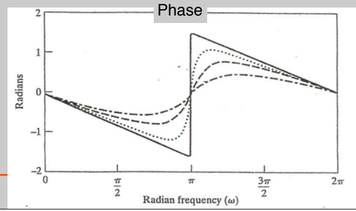
* Poles increase magnitude, but introduce phase lag and group delay.
* Zeros do the opposite.
* These effects are more marked when $r \to 1$.



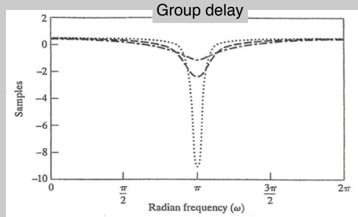
Log magnitude



Phase



Group delay

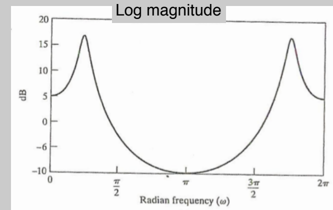


2nd order IIR example

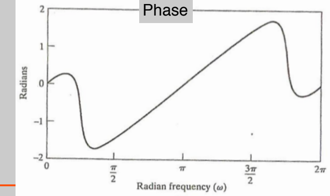
Example: 2nd order IIR with complex poles

$$H(z) = \frac{1}{(1 - re^{j\theta} z^{-1})(1 - re^{-j\theta} z^{-1})}$$

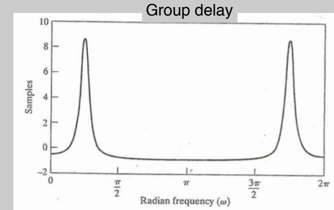
Log magnitude



Phase

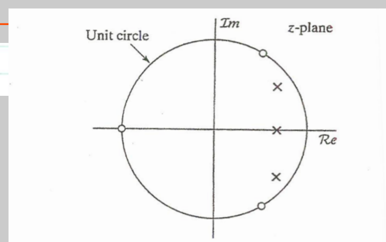


Group delay

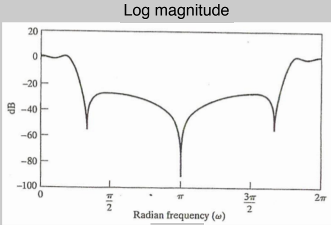


3nd order IIR example

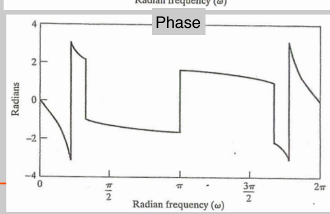
Example: 3rd order IIR



Log magnitude



Phase



Group delay