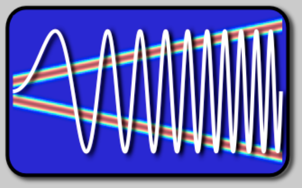

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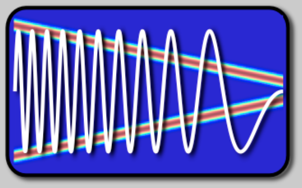
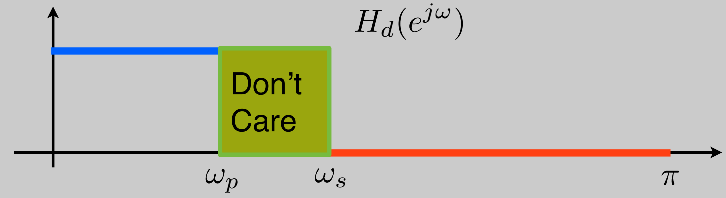


Digital Signal Processing

Lecture 23

Optimality



- Least Squares:

$$\text{minimize} \int_{\omega \in \text{care}} |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

Variation: weighted least-squares

$$\text{minimize} \int_{-\pi}^{\pi} W(\omega)|H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

Design Through Optimization

- Idea: Sample/discretize the frequency response

$$H(e^{j\omega}) \Rightarrow H(e^{j\omega_k})$$

– Sample points are fixed $\omega_k = k\frac{\pi}{P}$

$$-\pi \leq \omega_1 < \cdots < \omega_p \leq \pi$$

– M+1 is the filter order

– P >> M + 1 (rule of thumb P=15M)

– Yields a (good) approximation of the original problem

Example: Least Squares

- Target: Design M+1= 2N+1 filter
- First design non-causal $\tilde{H}(e^{j\omega})$ and hence $\tilde{h}[n]$
- Then, shift to make causal

$$h[n] = \tilde{h}[n - M/2]$$

$$H(e^{j\omega}) = e^{-j\frac{M}{2}}\tilde{H}(e^{j\omega})$$

Example: Least Squares

- Matrix formulation:

$$\tilde{h} = \left[\tilde{h}[-N], \tilde{h}[-N+1], \cdots, \tilde{h}[N]\right]^T$$

$$b = \left[H_d(e^{j\omega_1}), \cdots, H_d(e^{j\omega_P})\right]^T$$

$$A = \begin{bmatrix} e^{-j\omega_1(-N)} & \dots & e^{-j\omega_1(+N)} \\ \vdots & & \\ e^{-j\omega_P(-N)} & \dots & e^{-j\omega_P(+N)} \end{bmatrix}$$

$$\text{argmin}_{\tilde{h}} \quad ||A\tilde{h} - b||^2$$

Least Squares

$$\text{argmin}_{\tilde{h}} \quad ||A\tilde{h} - b||^2$$

Solution:

$$\tilde{h} = (A^*A)^{-1}A^*b$$

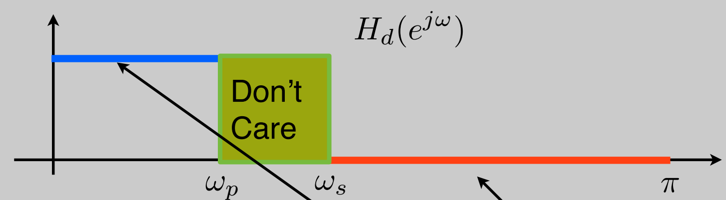
- Result will generally be non-symmetric and complex valued.
- However, if $\tilde{H}(e^{j\omega})$ is real, $\tilde{h}[n]$ should have symmetry!

Design of Linear-Phase L.P Filter

- Suppose:
  - $\tilde{H}(e^{j\omega})$ is real-symmetric
  - M is even (M+1 taps)
- Then:
  - $\tilde{h}[n]$ is real-symmetric around midpoint
- So:

$$\begin{aligned} \tilde{H}(e^{j\omega}) &= \tilde{h}[0] + \tilde{h}[1]e^{-j\omega} + \tilde{h}[-1]e^{+j\omega} \\ &\quad + \tilde{h}[2]e^{-j2\omega} + \tilde{h}[-2]e^{+j2\omega} \dots \\ &= \tilde{h}[0] + 2\cos(\omega)\tilde{h}[1] + 2\cos(2\omega)\tilde{h}[2] + \cdots \end{aligned}$$

Least-Squares Linear-Phase Filter



Given M, ω_P, ω_s find the best LS filter:

$$A = \begin{bmatrix} 1 & \cdots & 2\cos(\frac{M}{2}\omega_1) \\ \vdots & & \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_p) \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_s) \\ \vdots & & \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_P) \end{bmatrix} \qquad b = [1, 1, \cdots, 1, 0, 0, \cdots, 0]^T$$

Least-Squares Linear-Phase Filter

Given M, ω_P, ω_S find the best LS filter:

$$A = \begin{bmatrix} 1 & \cdots & 2\cos(\frac{M}{2}\omega_1) \\ \vdots & & \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_p) \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_s) \\ \vdots & & \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_P) \end{bmatrix} \qquad b = [1, 1, \cdots, 1, 0, 0, \cdots, 0]^T$$

$$\tilde{h}_+ = [\tilde{h}[0], \cdots, \tilde{h}[\frac{M}{2}]]^T = (A^*A)^{-1}A^*b$$

$$\tilde{h} = \begin{cases} \tilde{h}_+[n] & n \geq 0 \\ \tilde{h}_+[-n] & n < 0 \end{cases}$$

$$h[n] = \tilde{h}[n - M/2]$$

Extension:

- LS has no preference for pass band or stop band
- Use weighting of LS to change ratio

want to solve the discrete version of:

$$\text{minimize} \int_{-\pi}^{\pi} W(\omega)|H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

where $W(\omega)$ is δp in the pass band and δs in stop band

Similarly: $W(\omega)$ is 1 in the pass band and $\delta p/\delta s$ in stop band

Weighted Least-Squares

$$\text{argmin}_{\tilde{h}_+} \quad (A\tilde{h}_+ - b)^* W^2 (A\tilde{h}_+ - b)$$

Solution:

$$\tilde{h}_+ = (A^*W^2A)^{-1}W^2A^*b$$

$$W = \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & \ddots & & & \\ & & & \frac{\delta_p}{\delta_s} & & \\ & & & & \ddots & \\ 0 & & & & & \frac{\delta_p}{\delta_s} \end{bmatrix}$$

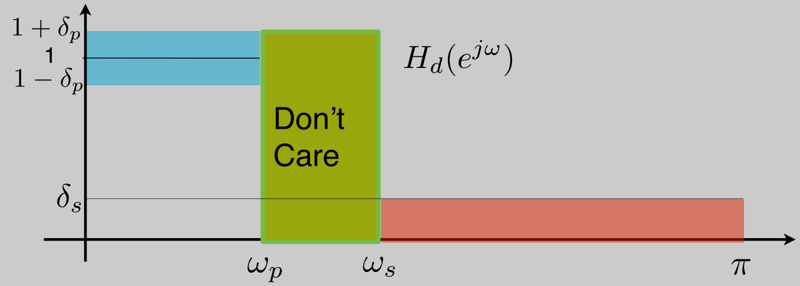
Min-Max optimal Filters

- Chebychev Design (min-max)

$$\text{minimize}_{\omega \in \text{care}} \quad \max |H(e^{j\omega}) - H_d(e^{j\omega})|$$

- – Parks-McClellan algorithm - equi-ripple
- – Also known as Remez exchange algorithms (signal.remez)
- –Also with convex optimization

Specifications



- Filter specifications are given in terms of boundaries

Min-Max Filter Design

- Minimize:
  - max pass-band ripple

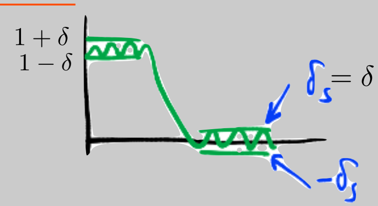
$$1-\delta_p \leq |H(e^{j\omega})| \leq 1+\delta_p, \qquad 0 \leq w \leq \omega_p$$

  - min-max stop-band ripple

$$|H(e^{j\omega})| \leq \delta_s, \qquad \omega_s \leq w \leq \pi$$

Min-max Ripple Design

- Recall, $\tilde{H}(e^{j\omega})$ is symmetric and real
- Given ωp ωs M, find δ,h̃₊:



$$\text{minimize} \quad \delta$$

Subject to :

$$1-\delta \leq \tilde{H}(e^{j\omega_k}) \leq 1+\delta \qquad 0 \leq \omega_k \leq \omega_p$$
$$-\delta \leq \tilde{H}(e^{j\omega_k}) \leq \delta \qquad \omega_s \leq \omega_k \leq \pi$$
$$\delta > 0$$

- Solution is a linear program in $\delta, \tilde{h}_+$
- A well studied class of problems

Min-Max Ripple via Linear Programming

$$\text{minimize} \quad \delta$$

subject to :

$$1-\delta \preceq A_p\tilde{h}_+ \preceq 1+\delta$$
$$-\delta \preceq A_s\tilde{h}_+ \preceq \delta$$
$$\delta > 0$$

$$A_p = \begin{bmatrix} 1 & 2\cos(\omega_1) & \cdots & 2\cos(\frac{M}{2}\omega_1) \\ & \vdots & & \\ 1 & 2\cos(\omega_p) & \cdots & 2\cos(\frac{M}{2}\omega_p) \end{bmatrix}$$

$$A_s = \begin{bmatrix} 1 & 2\cos(\omega_s) & \cdots & 2\cos(\frac{M}{2}\omega_1) \\ & \vdots & & \\ 1 & 2\cos(\omega_P) & \cdots & 2\cos(\frac{M}{2}\omega_P) \end{bmatrix}$$

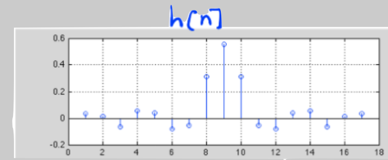
capital P

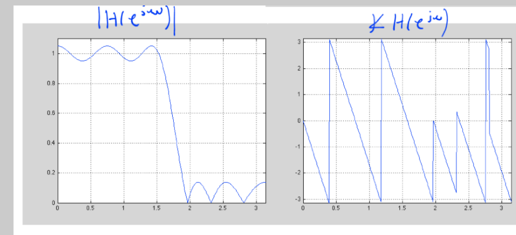
Convex Optimization

- Many tools and Solvers
- Tools:
  - –CVX (Matlab) http://cvxr.com/cvx/
  - – CVXOPT, CVXMOD (Python)
- Engines:
  - –Sedumi (Free)
  - – MOSEK (commercial)
- Take EE127!

Using CVX (in Matlab)

```
M = 16;
wp = 0.5*pi;
ws = 0.6*pi;
MM = 15*M;
w = linspace(0,pi,MM);

idxp = find(w <=wp);
idxs = find(w >=ws);

Ap = [ones(length(idxp),1) 2*cos(kron(w(idxp)',
[1:M/2]))];
As = [ones(length(idxs),1) 2*cos(kron(w(idxs)',
[1:M/2]))];

% optimization
cvx_begin
   variable hh(M/2+1,1);
   variable d(1,1);

   minimize(d)
   subject to
      Ap*hh <=1+d;
      Ap*hh >=1-d;
      As*hh <  d;
      As*hh > -d;
      ds>0;
cvx_end
h = [hh(end:-1:1) ; hh(2:end)];
```





Variations:

- Convex Problems:
  - –Fix δ_s optimize for δ_p
  - –Fix δ_p optimize for δ_s
  - –Linear constraints on h[n]
- Quasi-Convex (feasible through bisection)
  - –Fix δ_p, δ_s, M, minimize $\Delta\omega=\omega_s-\omega_p$
  - –Fix δ_p, δ_s, $\Delta\omega=\omega_s-\omega_p$, minimize M

Bisection Example: Minimize M

- given δ_p, δ_s, $\Delta\omega=\omega_s-\omega_p$ Initialize problem with:
  - – Set M_{min} to be small and hence infeasible
  - – Set M_{max} to be large and hence feasible
  - – Set M = floor(M_{max}/2 + M_{min}/2)
- Given M, δ_p, $\Delta\omega=\omega_s-\omega_p$ solve for minimum δ_s
  - – If δ_s violates constrains, set Mmin = M
  - – if δ_s within constraints, set Mmax = M
  - – Set M = floor(M_{max}/2 + M_{min}/2)
  - – Repeat till M is tight

IIR Design

- Historically
  - Continuous IIR design was advanced
  - Use results from C.T to D.T
  - C.T IIT designs have closed form, easy to use
  - Easy to control Magnitude, not easy to control phase
- Common Types:
  - Butterworth - monotonic, no ripple
  - Chebyshev - Type I, pass band ripple, Type II stop band ripple
  - Elliptic - Ripples in both bands

Design of D.T IIR Filters from Analog

- Discretize by one of many techniques
- Hc(s) $\Rightarrow$ H(z)
- Must satisfy:
  - Imaginary axis is mapped to unit circle
  - stability of Hc(s) should result in stable H(z)
- Two methods:
  - Impulse invariance - match impulse response
  - Bilinear transformation