EE123

Digital Signal Processing

Lecture 17

Carl Sagan - Cosmos

Billions

Topics

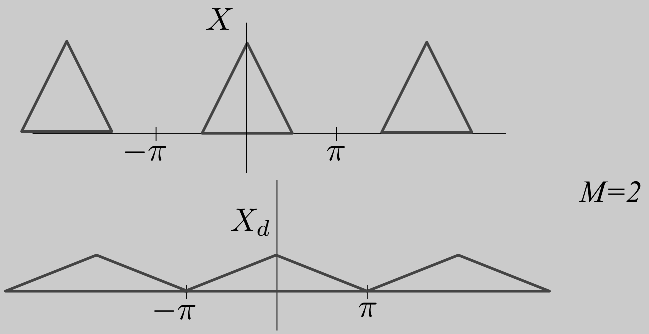
- Last time
  - Changing Sampling Rate via DSP
  - Downsampling
- Today
  - Upsampling

DownSampling

- Much like C/D conversion
- Expect similar effects:
  - Aliasing
  - mitigate by antialiasing filter
- Finely sampled signal ⇒ almost continuous
  - Downsample in that case is like sampling!

Changing Sampling-rate via D.T Processing

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0} X\left(e^{j(w/M-2\pi i/M)}\right)$$



Changing Sampling-rate via D.T Processing

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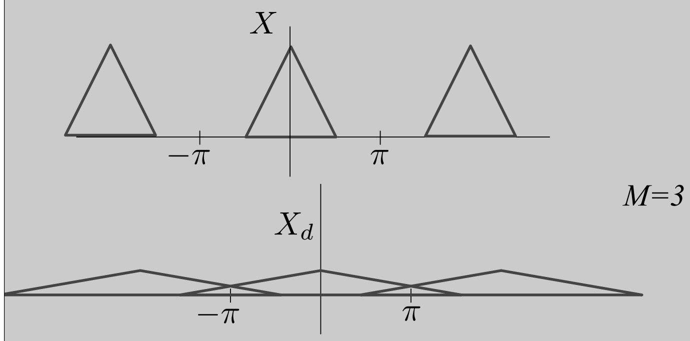
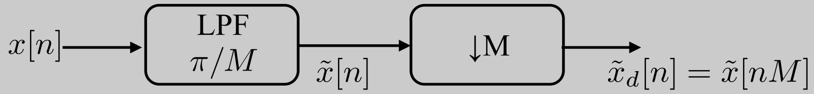
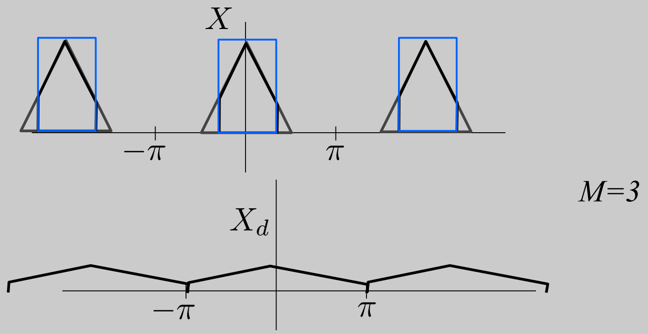


Anti-Aliasing





UpSampling

- Much like D/C converter
- Upsample by A LOT ⇒ almost continuous

- Intuition:
  - Recall our D/C model: x[n] ⇒ $x_s(t)$⇒$x_c(t)$
  - Approximate “$x_s(t)$” by placing zeros between samples
  - Convolve with a sinc to obtain “$x_c(t)$”

Up-sampling

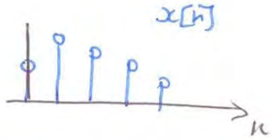
$x[n] = X_c(nT)$

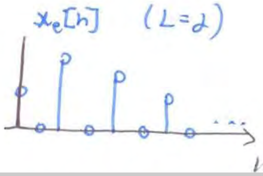
$x_i[n] = x_c(nT')$ where $T' = \frac{T}{L}$, L integer

obtain $x_i[n]$ from $x[n]$ in two steps:

(1) Generate $x_e = \begin{cases} x[n/L] & n = 0, \pm L, \pm 2L, \dots \\ 0 & \text{otherwise} \end{cases}$

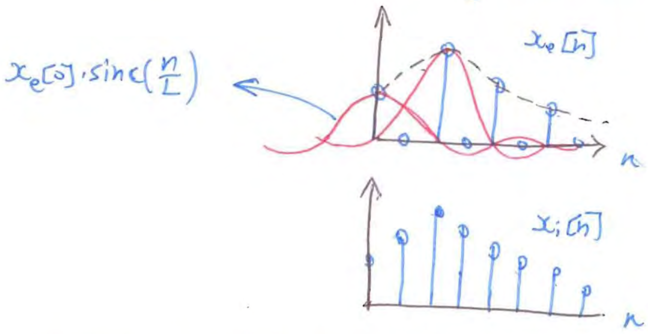
$\overset{\downarrow}{\text{expansion}}$





Up-Sampling

② Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation.



$x_i[n] = x_e[n] * \text{sinc}(\frac{n}{L})$

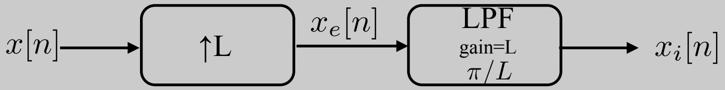
Up-Sampling

$$x_i[n] = x_e[n] * \text{sinc}(n/L)$$

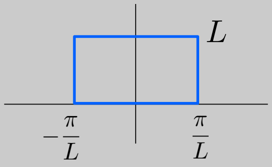
$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]$$

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k]\text{sinc}(\frac{n-kL}{L})$$

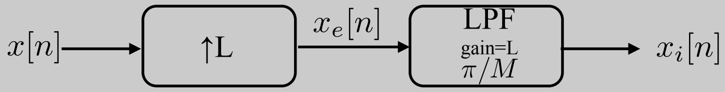
Frequency Domain Interpretation



$\text{sinc}(n/L)$ DTFT ⇒



Frequency Domain Interpretation

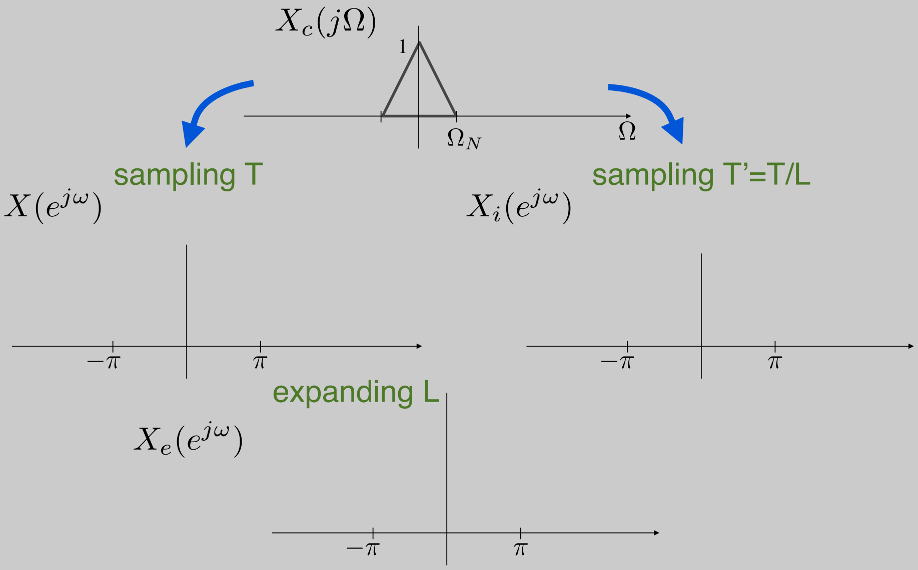


$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\neq 0 \text{ only for n=mL (integer m)}} e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL} = X(e^{j\omega L})$$

Compress DTFT by a factor of L!

Example:

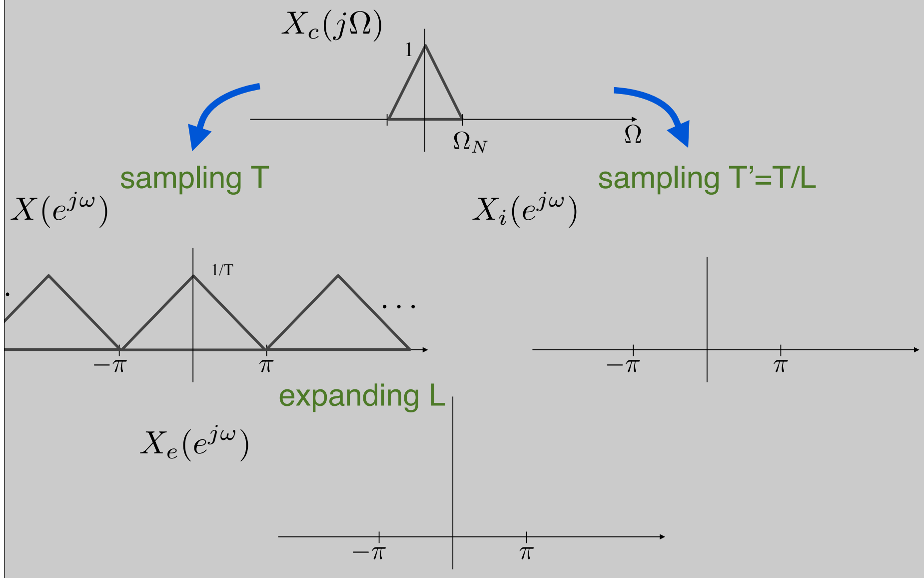


sampling T

sampling T'=T/L

expanding L

Example:



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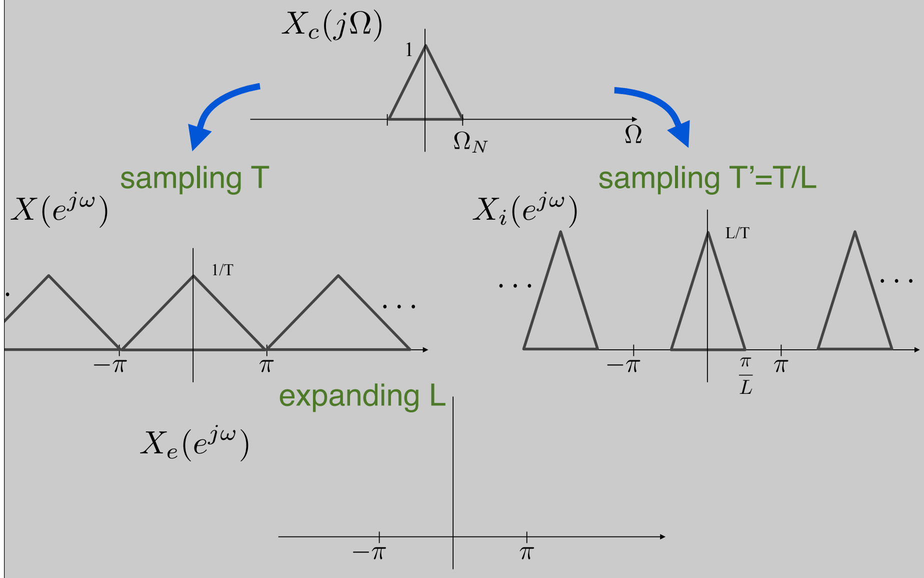

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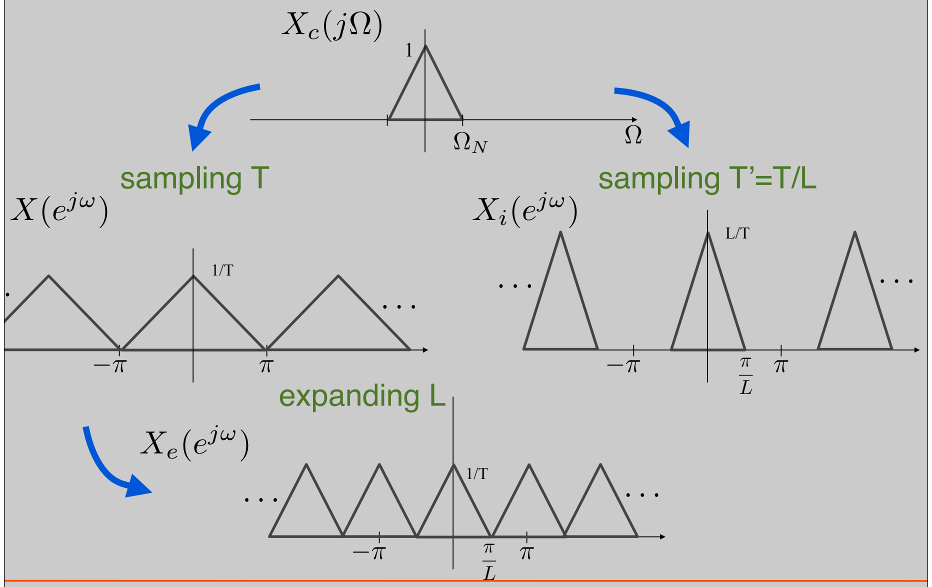

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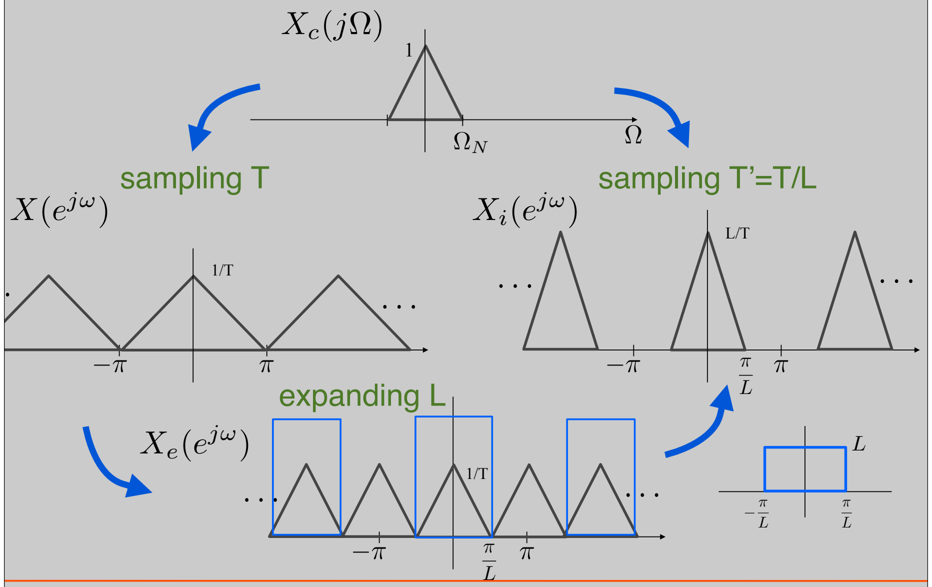
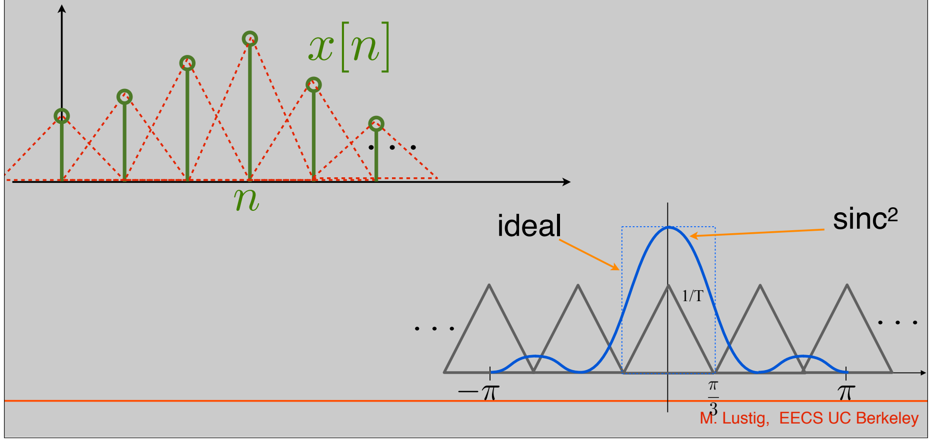


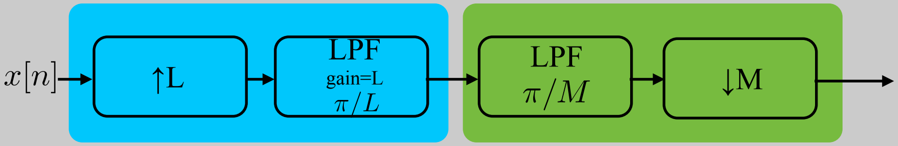
Practical Upsampling

- Can interpolate with simple, practical filters. What's happening?
- Example: L=3, linear interpolation - convolve with triangle

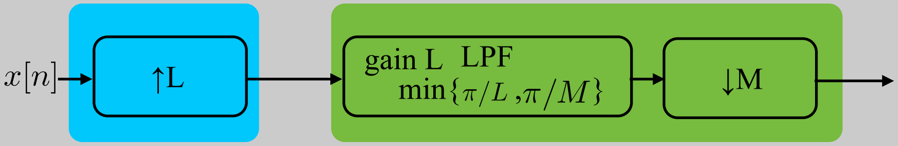


Resampling by non-integer

- T' = TM/L (upsample L, downsample M)



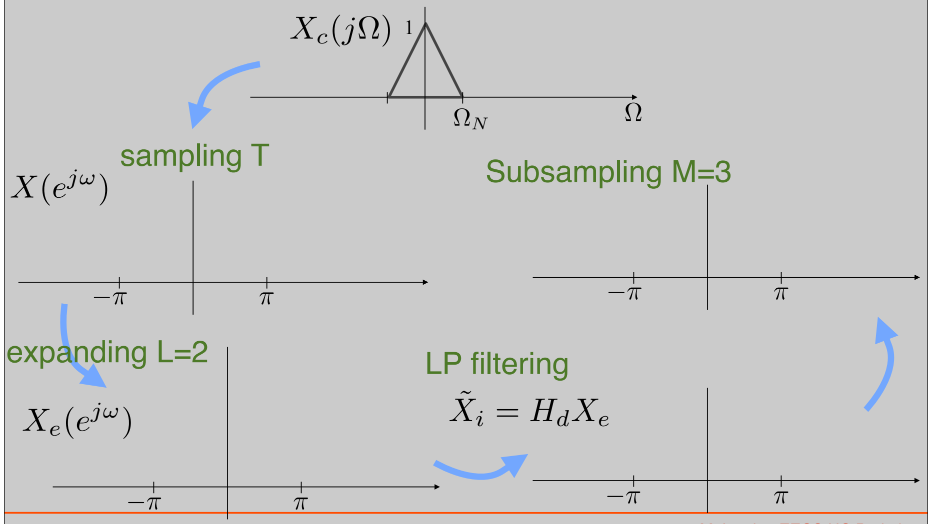
Or,



- What would happen if change order?

Example:

- L = 2, M=3, T'=3/2T (fig 4.30)



sampling T

Subsampling M=3

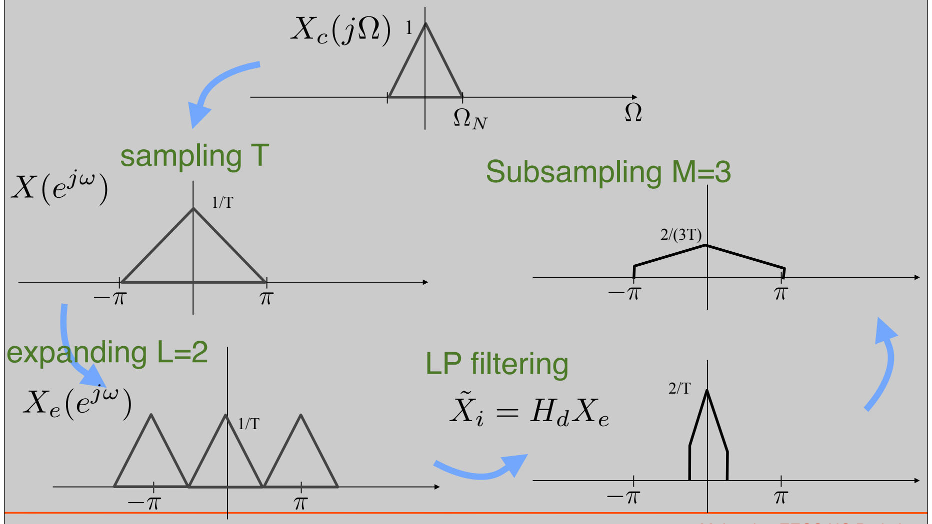
expanding L=2

LP filtering

$\tilde{X}_i = H_d X_e$

Example:

- L = 2, M=3, T'=3/2T (fig 4.30)



sampling T

Subsampling M=3

expanding L=2

LP filtering

$\tilde{X}_i = H_d X_e$

Multi-Rate Signal Processing

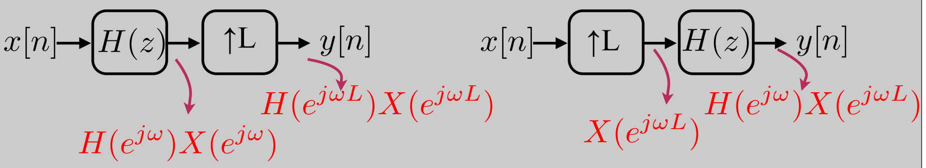
- What if we want to resample by 1.01T?
  - Expand by L=100
  - Filter π/101 ($$$$$)
  - Downsample by M=101
- Fortunately there are ways around it!
  - Called multi-rate
  - Uses compressors, expanders and filtering

Interchanging Operations

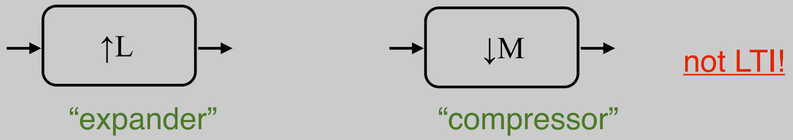
“expander” “compressor”

not LTI!

Note:

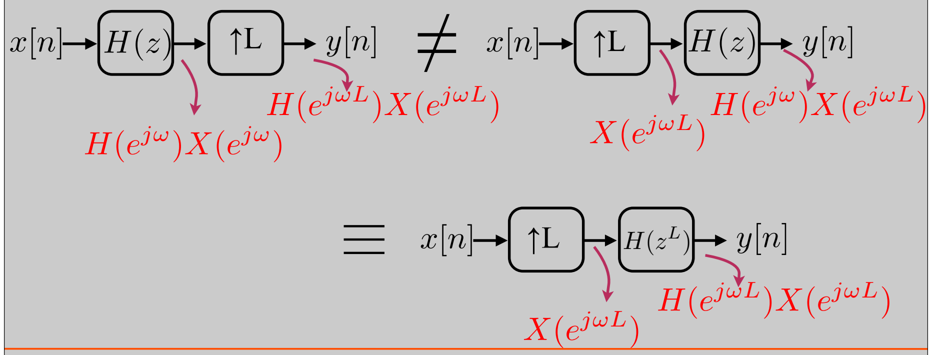


Interchanging Operations



“expander” “compressor” not LTI!

Note:

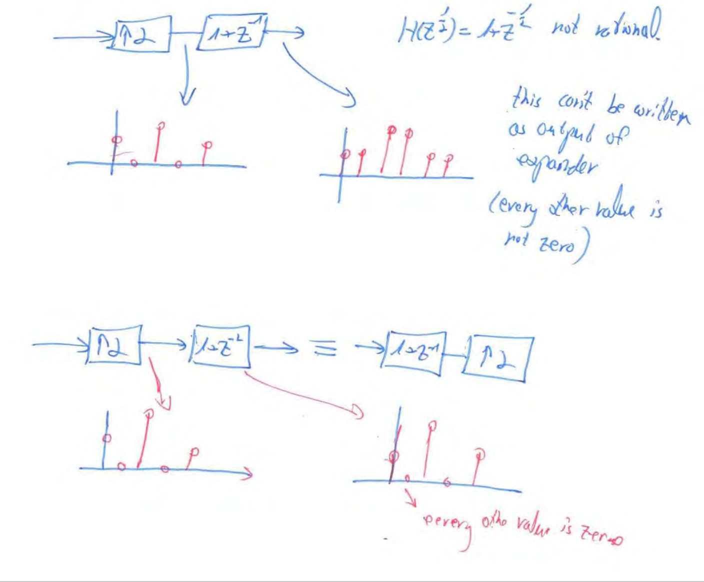


Interchanging Filter Expander

- Q: Can we move expander from Left to Right (with xform)?

- A: Yes, if H(z) is rational
  No, otherwise

Example:



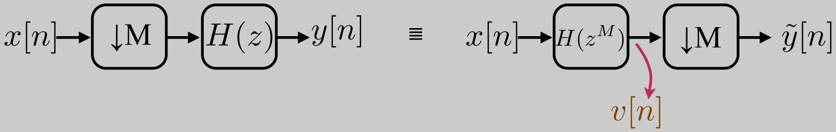
$H(z^{\frac{1}{2}}) = 1+z^{-\frac{1}{2}}$ not rational

this can't be written as output of expander (every other value is not zero)

every other value is zero

Compressor

Claim:



Proof:

Compressor

Proof:

$$Y(e^{j\omega}) = H(e^{j\omega})\left(\frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})})\right) =$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} \underbrace{H(e^{j(\omega-2\pi i)})}_{=H(e^{j\omega})} X(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})})$$

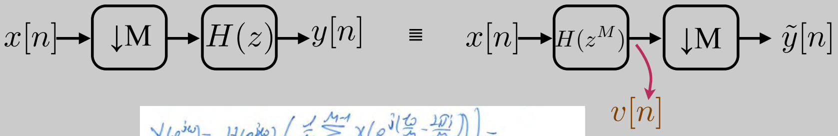
$$= \frac{1}{M}\sum_{i=0}^{M-1} H(e^{jM(\frac{\omega}{M}-\frac{2\pi i}{M})}) X(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})})$$

$$V(e^{j\omega}) = H(e^{j\omega M}) X(e^{j\omega})$$

after compressor

Compressor

Claim:



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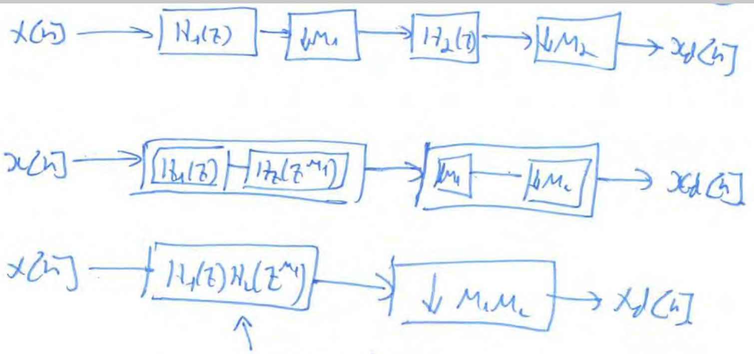
$$V(e^{j\omega}) = H(e^{j\omega M}) X(e^{j\omega})$$

after compressor

Q: Move compressor from right to left?

A: only if $H(z^{1/M})$ rational.

Multi-Rate Filtering



Very sharp ~~filter~~ Narrow band → long impulse response.

faster to do 2 stage.