EE123
Digital Signal Processing

Lecture 11

Last Time

- Started with STFT
- Heisenberg Boxes
- Continue and move to wavelets
- Ham -- Get me the forms!

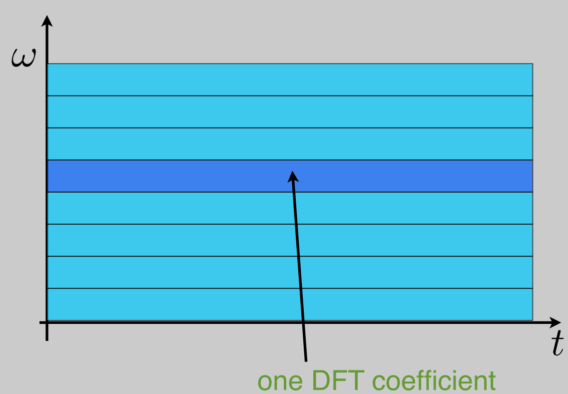
DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

$$\Delta\omega = \frac{2\pi}{N}$$

$$\Delta t = N$$

$$\Delta\omega \cdot \Delta t = 2\pi$$

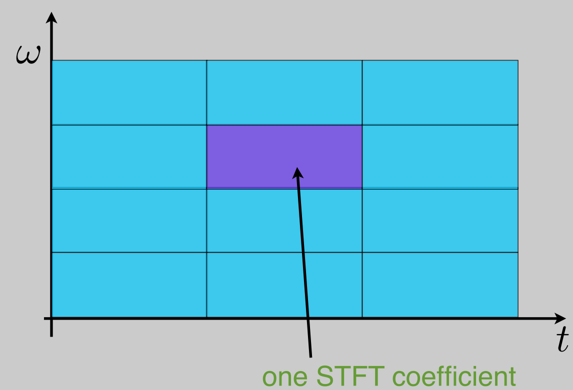


Discrete STFT

$$X[r,k] = \sum_{m=0}^{L-1} x[r\underbrace{R}_{\text{optional}} + m]w[m]e^{-j2\pi km/N}$$

$$\Delta\omega = \frac{2\pi}{L}$$

$$\Delta t = L$$



Limitations of Discrete STFT

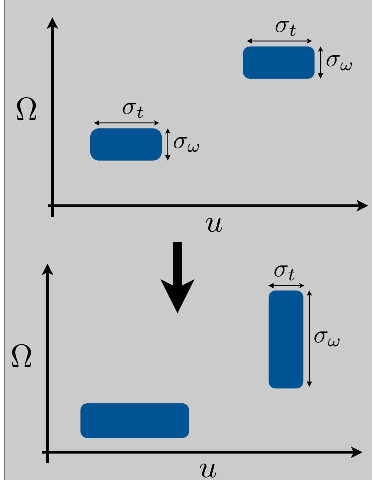
- Need overlapping ⇒ Not orthogonal
- Computationally intensive O(MN log N)
- Same size Heisenberg boxes

From STFT to Wavelets

- Basic Idea:
  – low-freq changes slowly - fast tracking unimportant
  – Fast tracking of high-freq is important in many apps.
  – Must adapt Heisenberg box to frequency
- Back to continuous time for a bit.....

From STFT to Wavelets

- Continuous time



$$Sf(u,\Omega) = \int_{-\infty}^{\infty} f(t)w(t-u)e^{-j\Omega t}dt$$

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\Psi^*(\frac{t-u}{s})dt$$

*Morlet - Grossmann

From STFT to Wavelets

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\Psi^*(\frac{t-u}{s})dt$$

- The function Ψ is called a mother wavelet
  – Must satisfy:

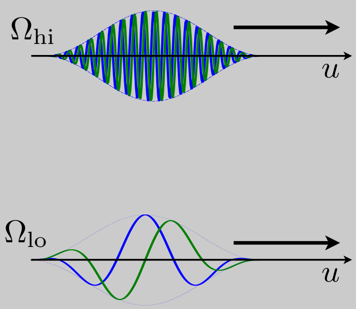
$$\int_{-\infty}^{\infty} |\Psi(t)|^2 dt = 1 \quad \Rightarrow \text{unit norm}$$

$$\int_{-\infty}^{\infty} \Psi(t)dt = 0 \quad \Rightarrow \text{Band-Pass}$$

STFT and Wavelets “Atoms”

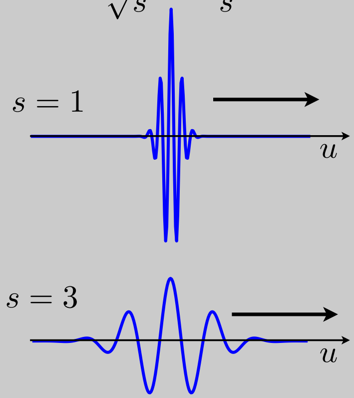
STFT Atoms
(with hamming window)

$$w(t-u)e^{j\Omega t}$$



Wavelet Atoms

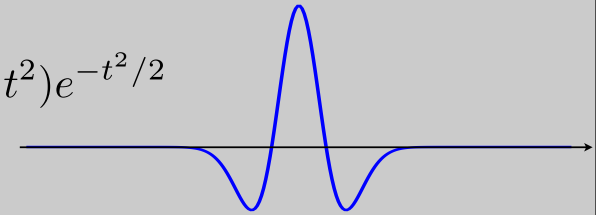
$$\frac{1}{\sqrt{s}}\Psi(\frac{t-u}{s})$$



Examples of Wavelets

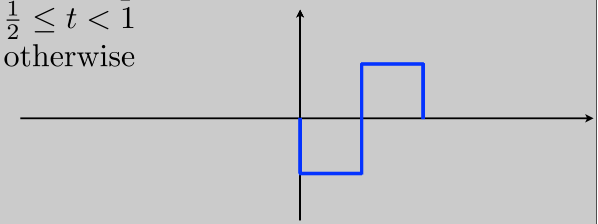
- Mexican Hat

$$\Psi(t) = (1-t^2)e^{-t^2/2}$$

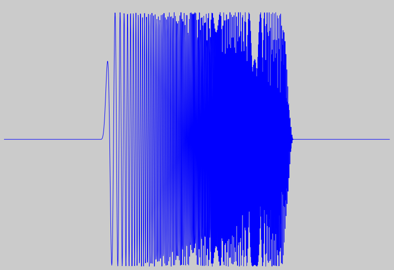


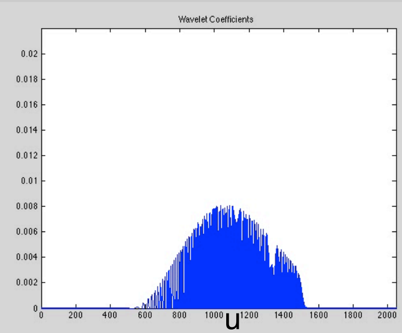
- Haar

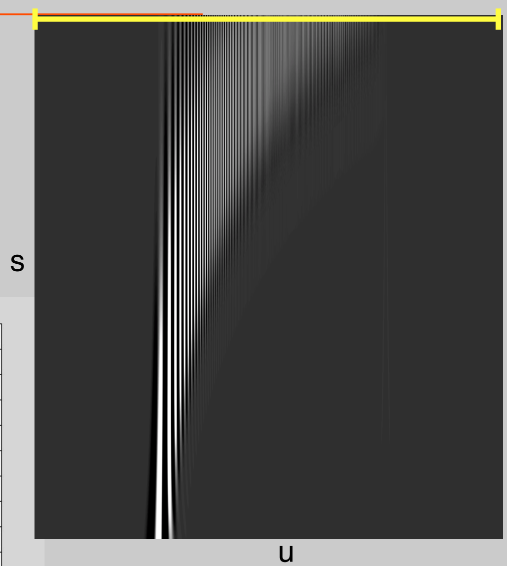
$$\Psi(t) = \begin{cases} -1 & 0 \le t < \frac{1}{2} \\ 1 & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$



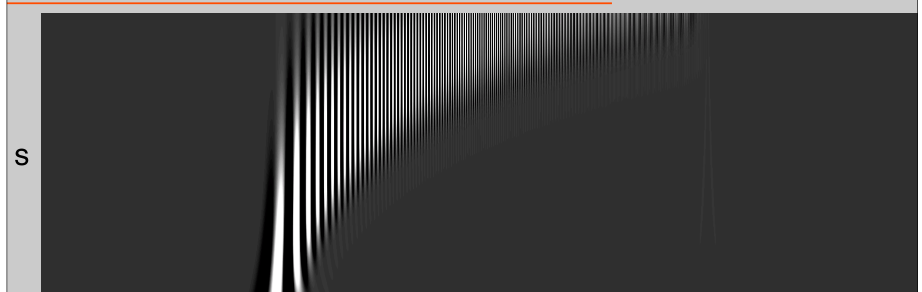
Example: Wavelet of Chirp

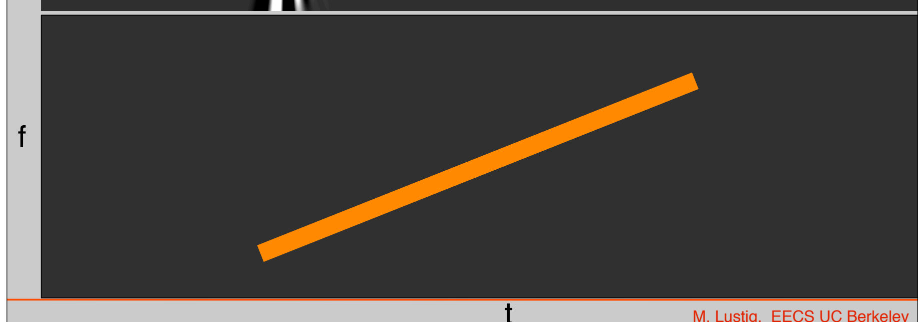




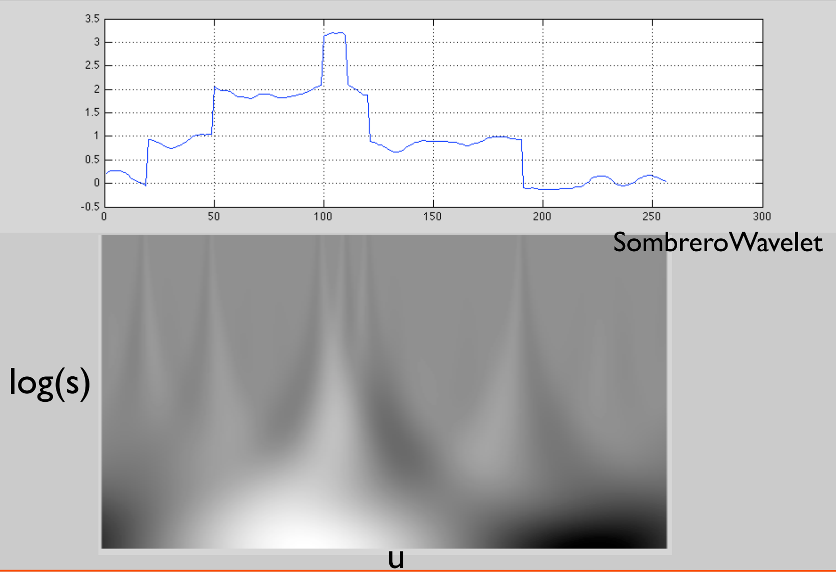


Wavelets VS STFT





Example 2: “Bumpy” Signal



Wavelets Transform

- Can be written as linear filtering

$$
\begin{aligned}
Wf(u,s) &= \frac{1}{\sqrt{s}}\int_{-\infty}^{\infty} f(t)\Psi^*\left(\frac{t-u}{s}\right)dt \\
&= \left\{f(t) * \overline{\Psi}_s(t)\right\}(u)
\end{aligned}
$$

$$\overline{\Psi}_s = \frac{1}{\sqrt{s}}\Psi\left(\frac{t}{s}\right)$$

- Wavelet coefficients are a result of bandpass filtering

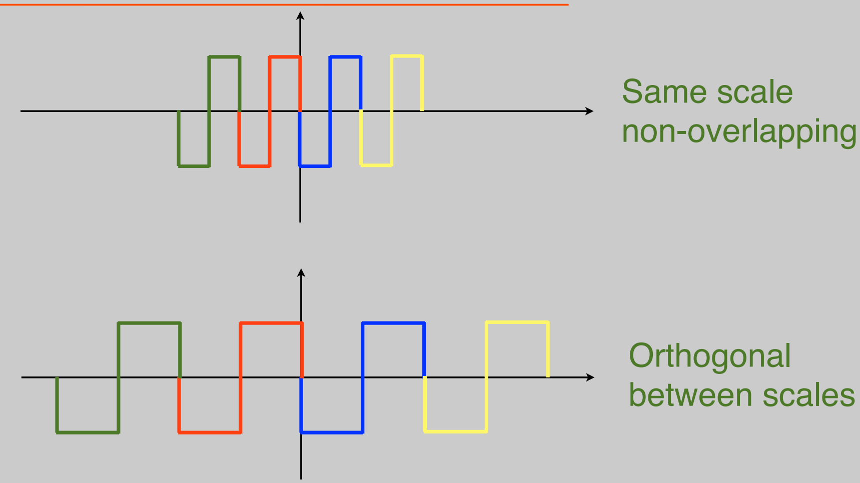
Wavelet Transform

- Many different constructions for different signals
  - Haar good for piece-wise constant signals
  - Battle-Lemarie’ : Spline polynomials
- Can construct Orthogonal wavelets
  - For example: dyadic Haar is orthonormal

$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi\left(\frac{t-2^i n}{2^i}\right)$$

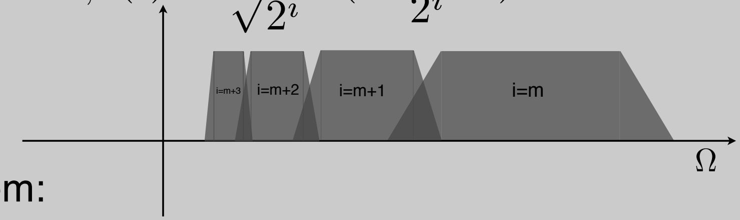
$$i = [1,\ 2,\ 3, \cdots]$$

Orthonormal Haar



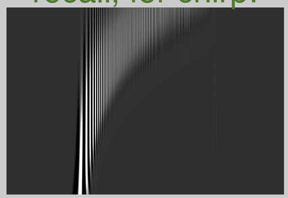
Scaling function

$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi\left(\frac{t-2^i n}{2^i}\right)$$



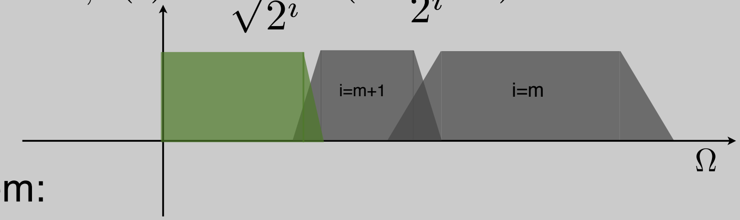
- Problem:
  - Every stretch only covers half remaining bandwidth
  - Need Infinite functions

recall, for chirp:



Scaling function

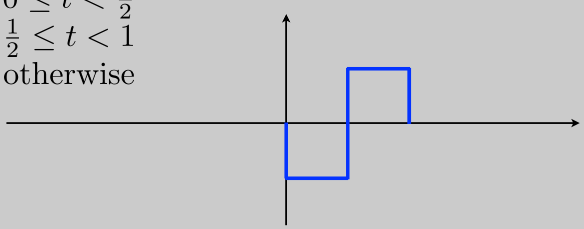
$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi\left(\frac{t-2^i n}{2^i}\right)$$



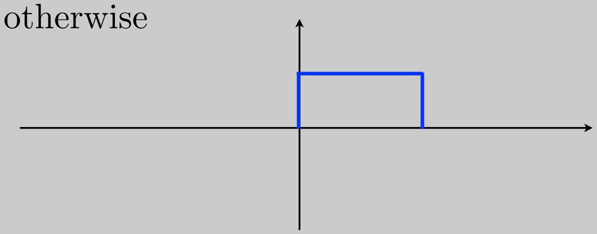
- Problem:
  - Every stretch only covers half remaining bandwidth
  - Need Infinite functions
- Solution:
  - Plug low-pass spectrum with a scaling function $\overline{\Phi}$

Haar Scaling function

$$\Psi(t) = \begin{cases} -1 & 0 \le t < \frac{1}{2} \\ 1 & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\Phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$



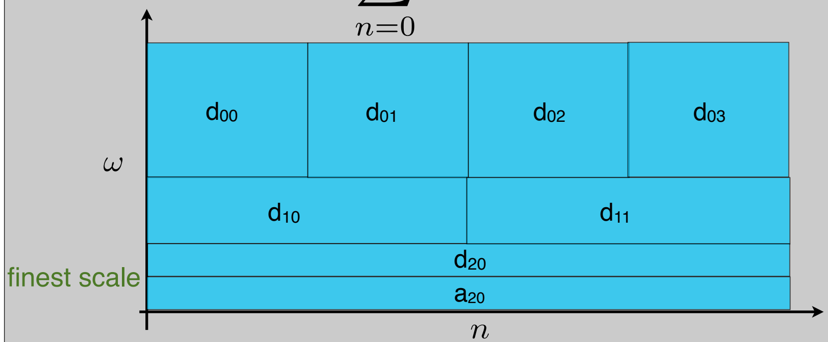
Back to Discrete

- Early 80's, theoretical work by Morlett, Grossman and Meyer (math, geophysics)
- Late 80's link to DSP by Daubechies and Mallat.
- From CWT to DWT not so trivial!
- Must take care to maintain properties

Discrete Wavelet Transform

$$d_{\mathrm{s,u}} = \sum_{n=0}^{N-1} x[n]\Psi_{\mathrm{s,u}}[n]$$

$$a_{\mathrm{s,u}} = \sum_{n=0}^{N-1} x[n]\Phi_{\mathrm{s,u}}[n]$$



Discrete Wavelet Transform

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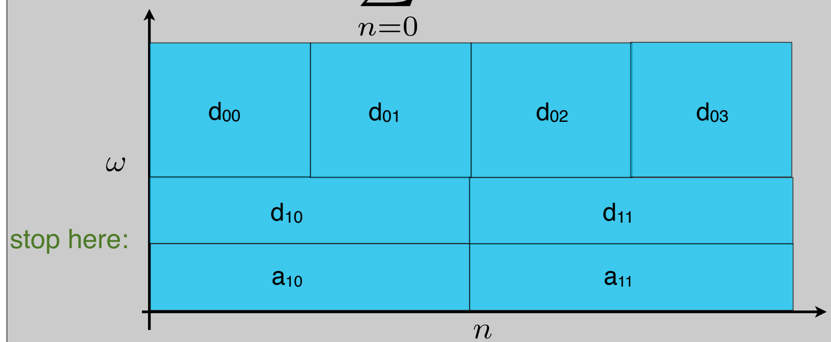
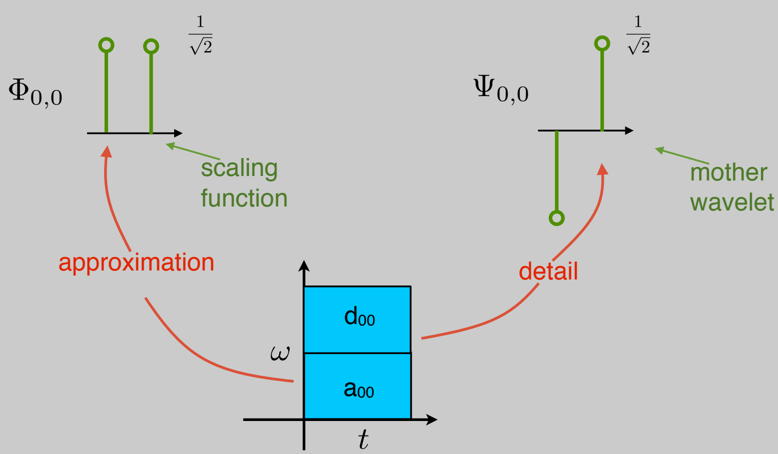


Example: Discrete Haar Wavelet

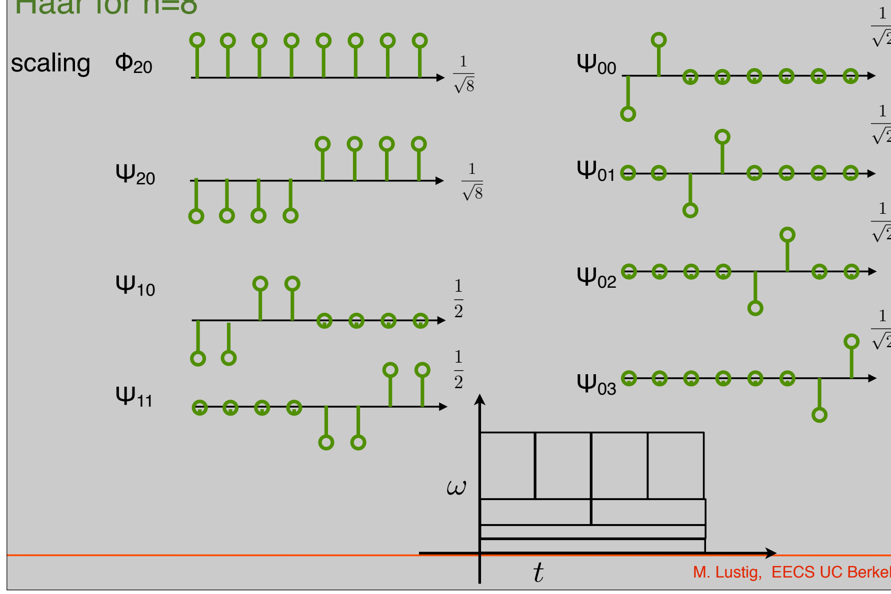
Haar for n=2



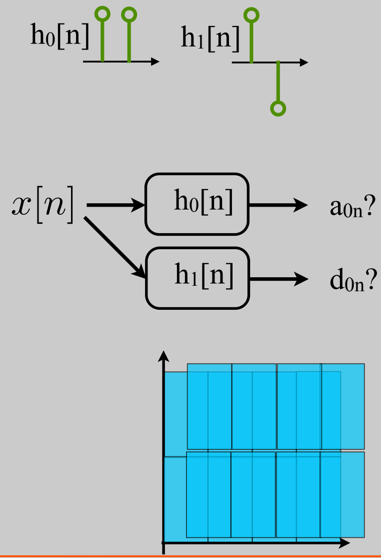
Equivalent to DFT_2!

Discrete Orthogonal Haar Wavelet

Haar for n=8

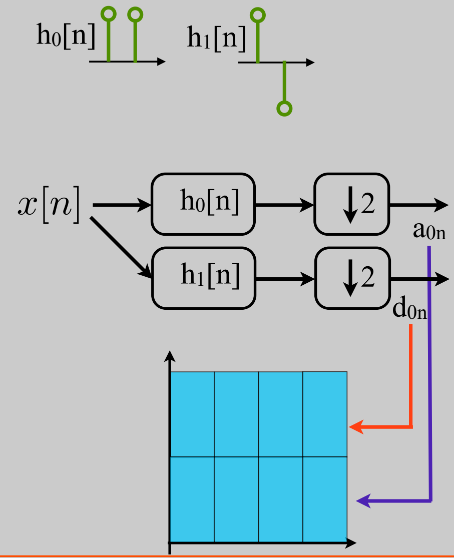


Fast DWT with Filter Banks (more Later!)

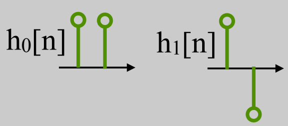


not quite...
too many
coefficients

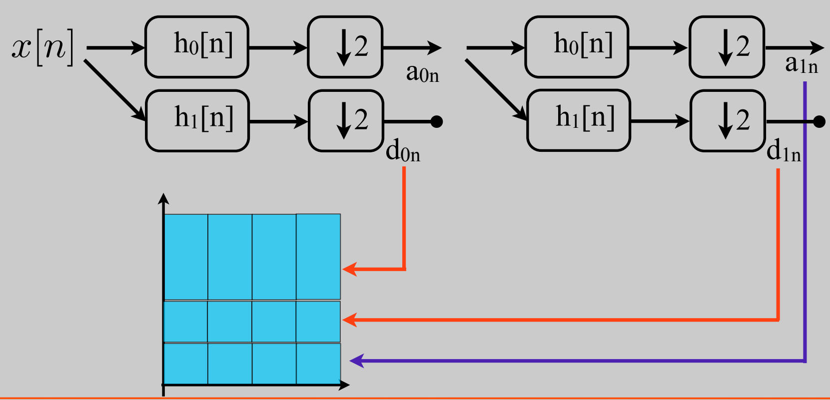
Fast DWT with Filter Banks



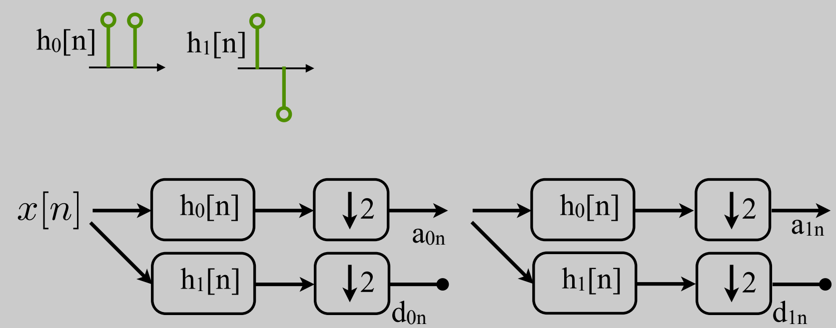
Fast DWT with Filter Banks



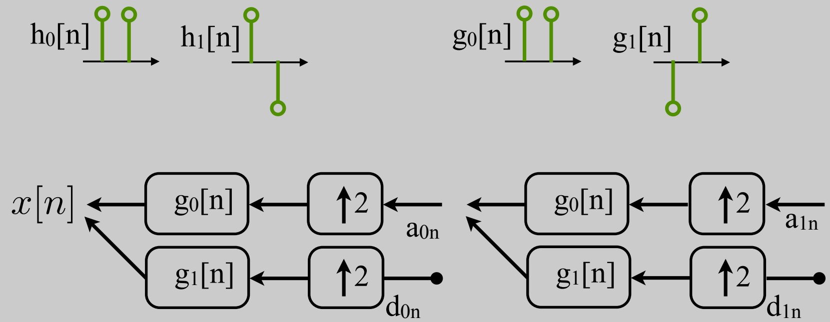
complexity:
N + N/2 + N/4 + N/8 +...+ = 2N
=O(N)



Decomposition

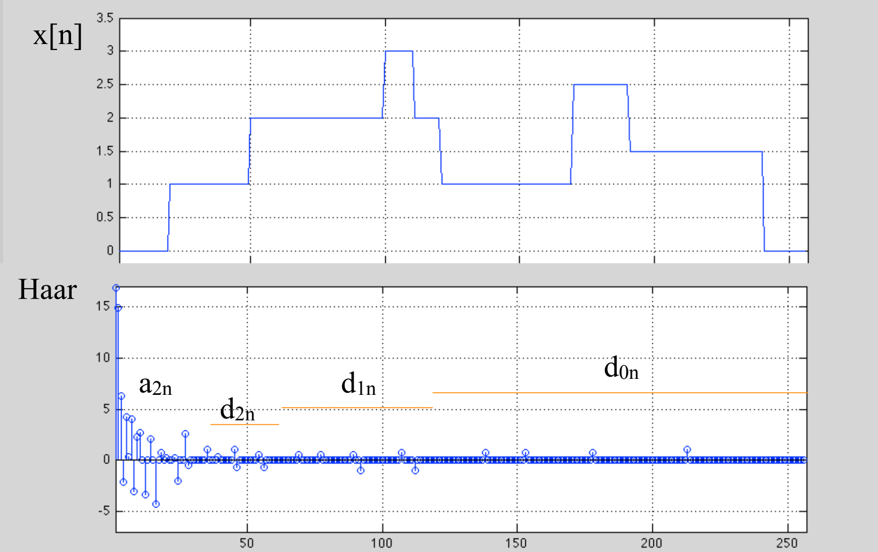


Reconstruction

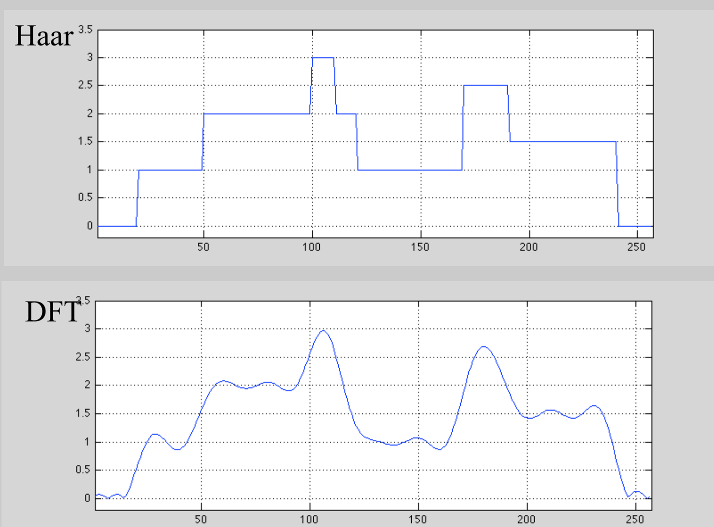


Just flip arrows,
replace h with g

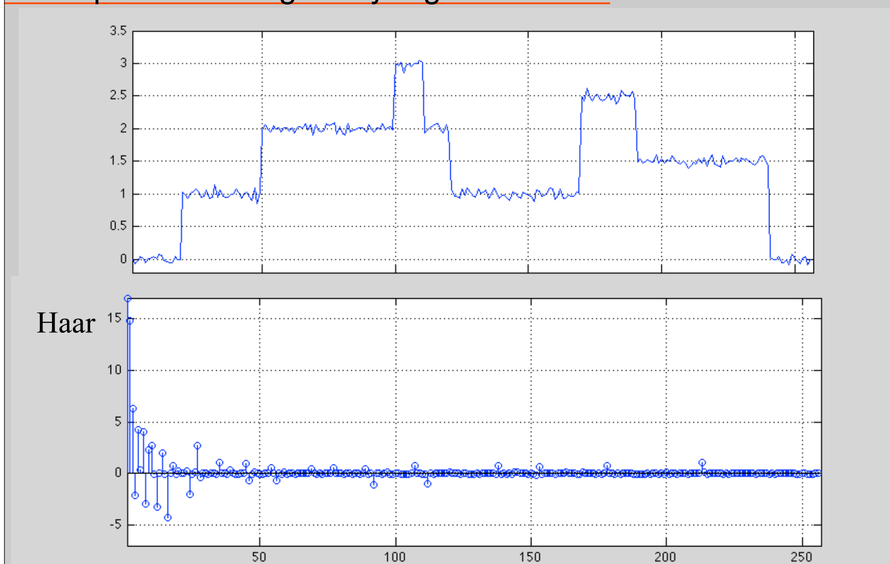
Haar DWT Example



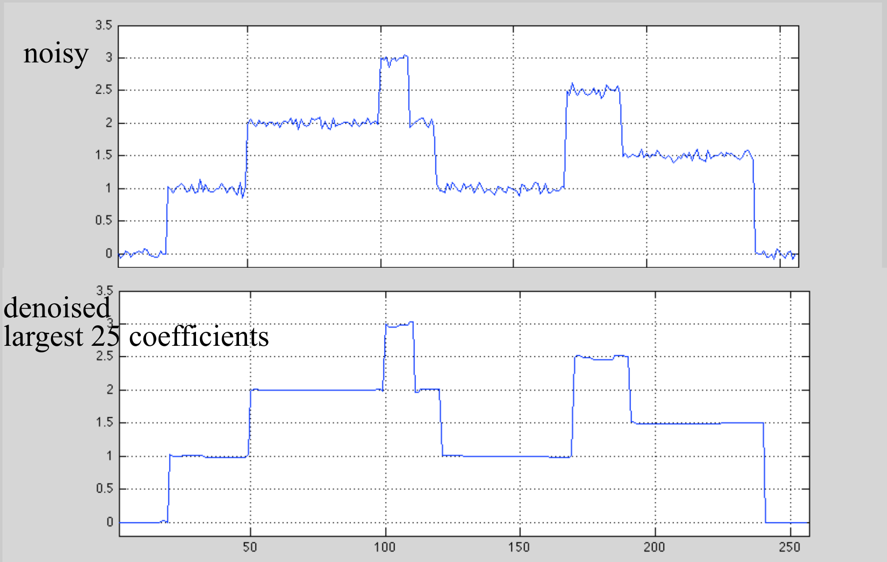
Approximation from 25/256 coefficients



Example: Denoising Noisy Signals



Example: Denoising by Thresholding



Compression - JPEG2000 vs JPEG

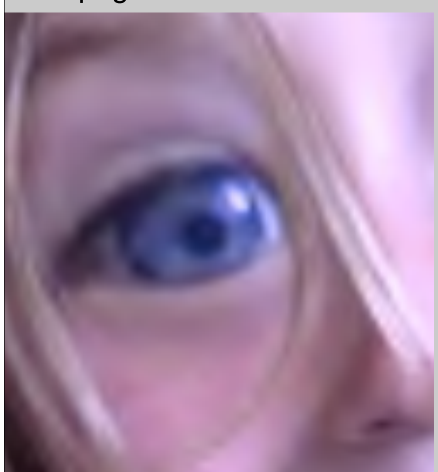
Jpeg2000 - Wavelet

Jpeg - DCT

@ 66 fold compression ratio

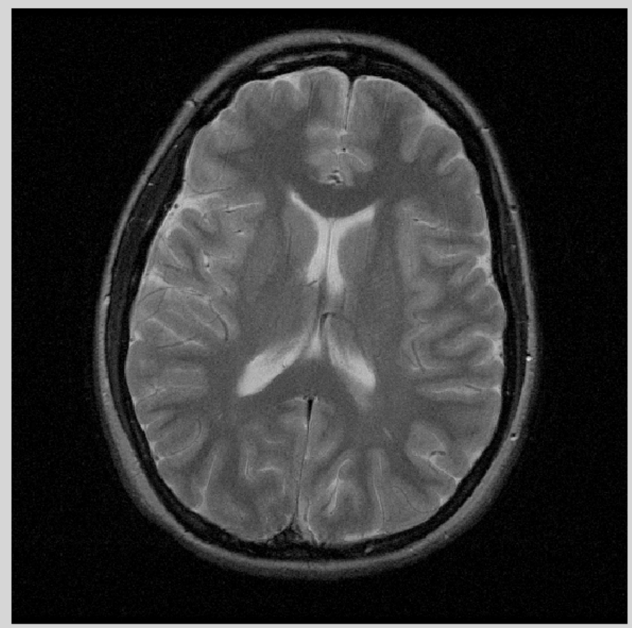
Compression - JPEG2000 vs JPEG

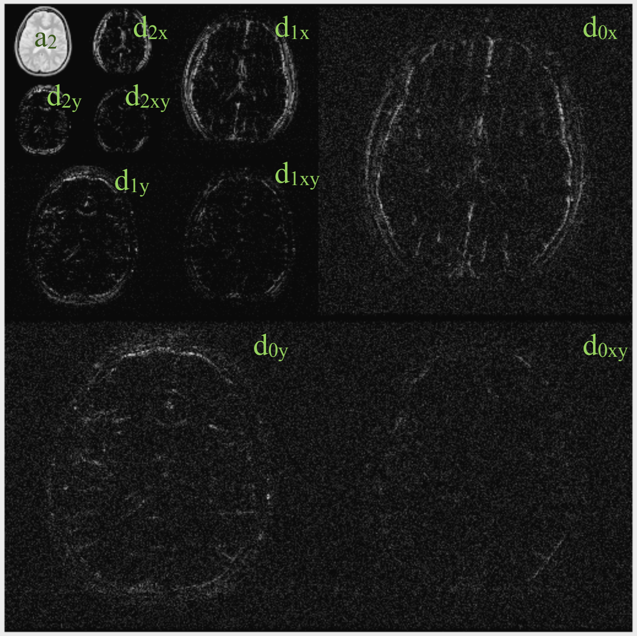
Jpeg2000 - Wavelet

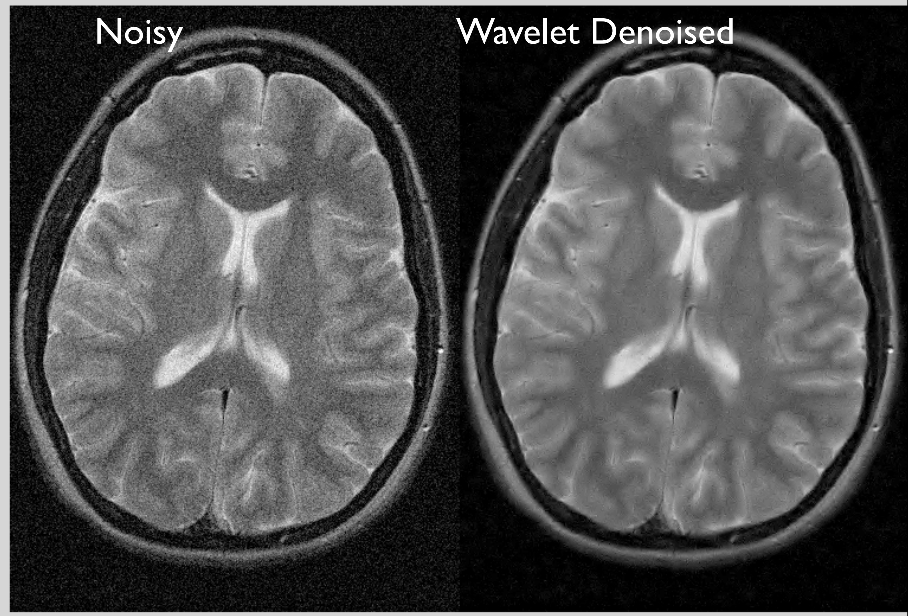


Jpeg - DCT

@ 66 fold compression ratio

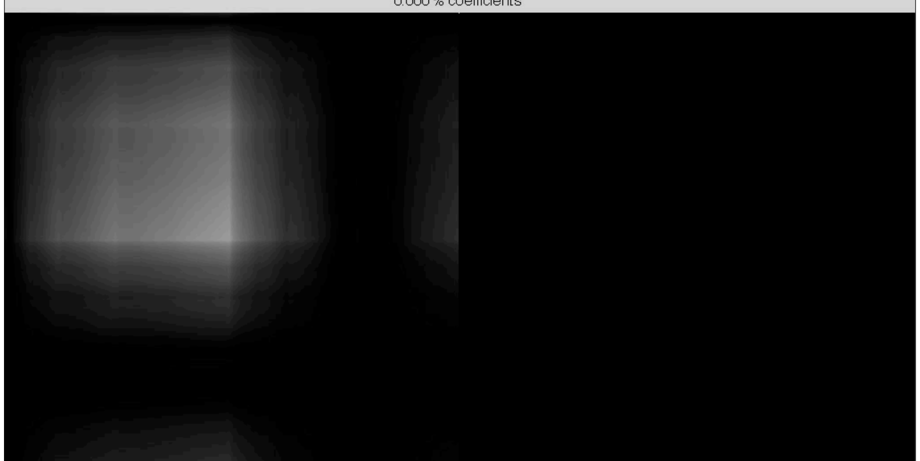






Approximation/Compression

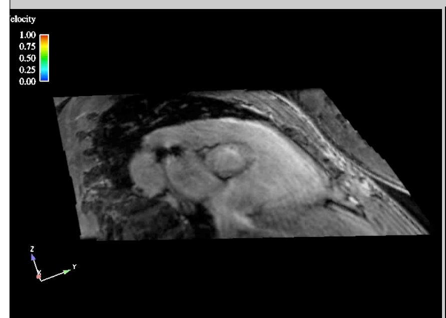
0.000 % coefficients

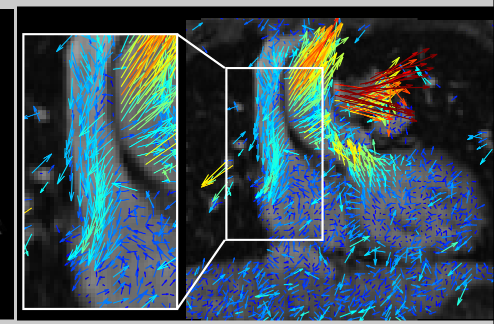


Example in Research

Robust 4D Flow Denoising using Divergence-free Wavelet Transform

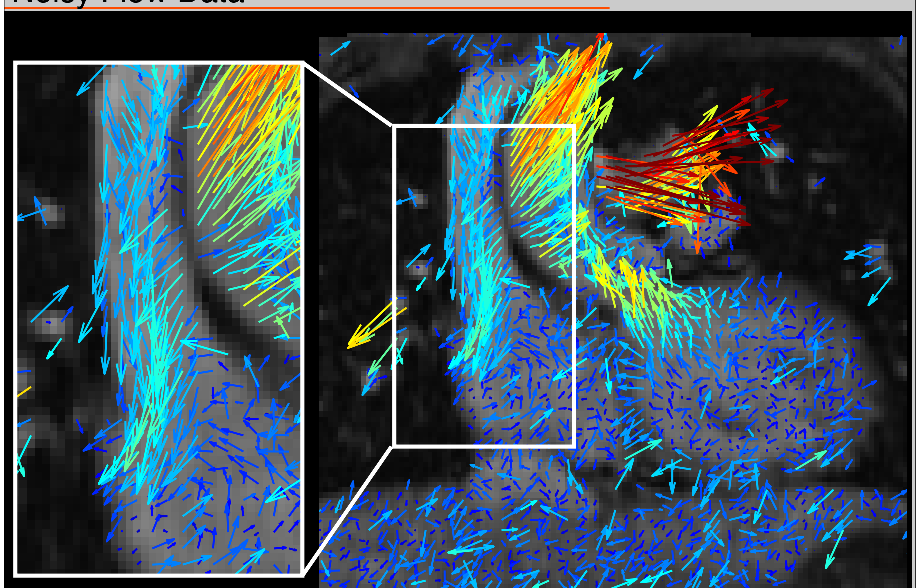
Frank Ong[1], Martin Uecker[1], Umar Tariq[2], Albert Hsiao[2], Marcus T Alley[2],
Shreyas S Vasanawala[2], Michael Lustig[1]



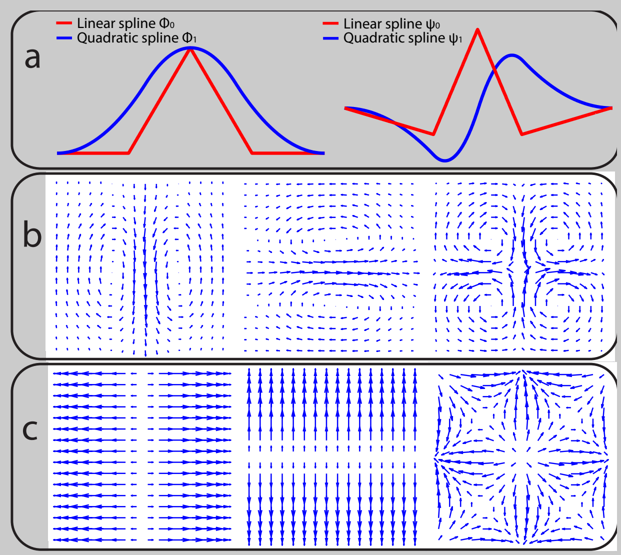


courtesy, Frank Ong and Marcus Alley

Noisy Flow Data



Divergence Free Wavelets



Divergence-Free Wavelet Denoising