EE123
Digital Signal Processing

Lecture 5

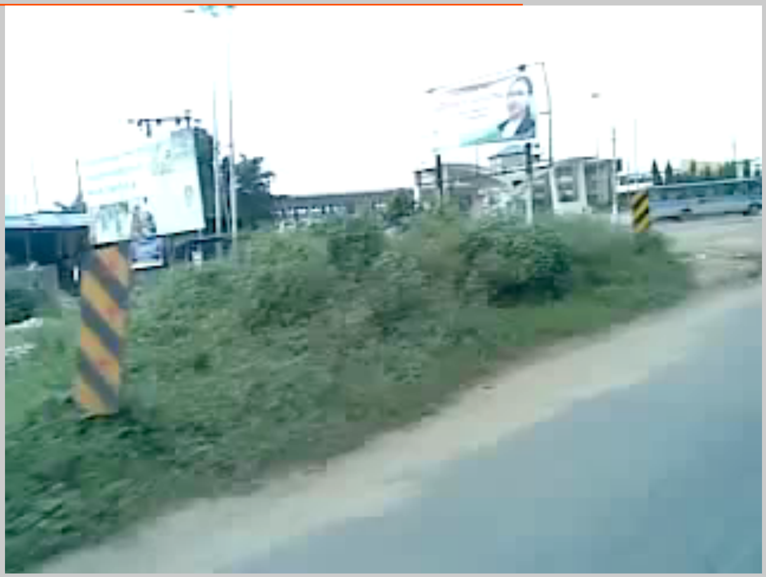
based on slides by J.M. Kahn

Info

- Last time
  - Finished DTFT Ch. 2
  - 12min z-Transforms Ch. 3
- Today: DFT Ch. 8

- Reminders:
  - HW Due tonight
  - Ham lecture 5-6pm HP auditorium

What is this Phenomena?



Motivation: Discrete Fourier Transform

- Sampled Representation in time and frequency
  - Numerical Fourier Analysis requires discrete representation
  - But, sampling in one domain corresponds to periodicity in the other...
  - What about DFS (DFT)?
    - Periodic in "time" ✓
    - Periodic in "Frequency" ✓
  - What about non-periodic signals?
    - Still use DFS(T), but need special considerations

Motivation: Discrete Fourier Transform

- Efficient Implementations exist
  - Direct evaluation of DFT: $O(N^2)$
  - Fast Fourier Transform (FFT): O(N log N)
    (ch. 9, next topic....)
  - Efficient libraries exist: FFTW
    - In Python:
      > X = np.fft.fft(x);
      > x = np.fft.ifft(X);
  - Convolution can be implemented efficiently using FFT
    - Direct convolution: $O(N^2)$
    - FFT-based convolution: O(N log N)

Discrete Fourier Series (DFS)

- Definition:
  - Consider N-periodic signal:

$$\tilde{x}[n+N] = \tilde{x}[n] \quad \forall n$$

frequency-domain N-periodic representation:

$$\tilde{X}[k+N] = \tilde{X}[k] \quad \forall k$$

  - "~" indicates periodic signal/spectrum

Discrete Fourier Series (DFS)

- Define:

$$W_N \triangleq e^{-j2\pi/N}$$

- DFS:

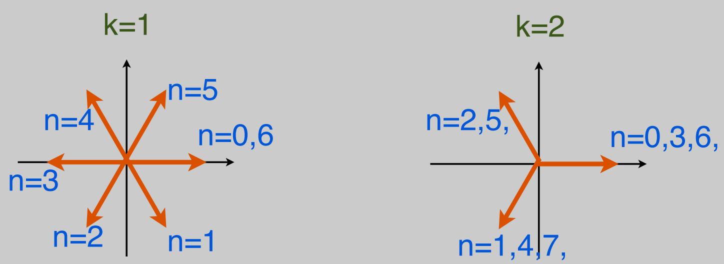
$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

Properties of W_N^{kn}?

Discrete Fourier Series (DFS)

- Properties of W_N:
  - $W_N^0 = W_N^N = W_N^{2N} = \ldots = 1$
  - $W_N^{k+r} = W_N^k W_N^r$ or, $W_N^{k+N} = W_N^k$
- Example: W_N^{kn} (N=6)



Discrete Fourier Transform

- By Convention, work with **one** period:

$$x[n] \triangleq \begin{cases} \tilde{x}[n] & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X[k] \triangleq \begin{cases} \tilde{X}[k] & 0 \leq k \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

Same same..... but different!

Discrete Fourier Transform

- The DFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_n^{-kn}$$ Inverse DFT, synthesis

$$X[k] = \sum_{n=0}^{N-1} x[n] W_n^{kn}$$ DFT, analysis

- It is understood that,

$$x[n] = 0 \quad \text{outside } 0 \leq n \leq N-1$$
$$X[k] = 0 \quad \text{outside } 0 \leq k \leq N-1$$

Discrete Fourier Transform

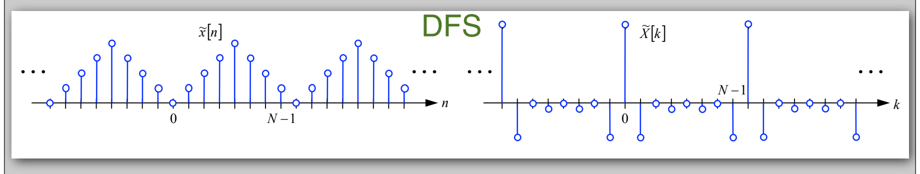
- Alternative formulation (not in book) Orthonormal DFT:

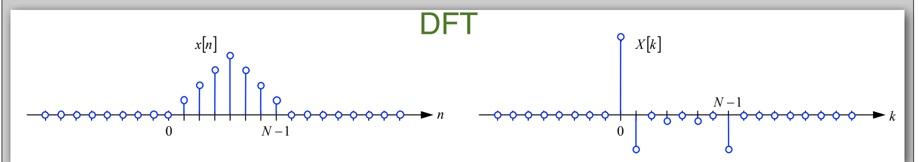
$$x[n] = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} X[k] W_n^{-kn}$$ Inverse DFT, synthesis

$$X[k] = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x[n] W_n^{kn}$$ DFT, analysis

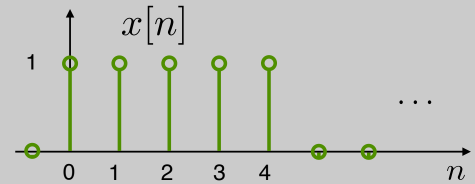
Why use this or the other?

Comparison between DFS/DFT





Example



- Take N=5

$$X[k] = \begin{cases} \sum_{n=0}^{4} W_5^{nk} & k = 0,1,2,3,4 \\ 0 & \text{otherwise} \end{cases}$$
$$= 5\delta[k]$$

"5-point DFT"

Example

- Q: What if we take N=10?

A: $X[k] = \tilde{X}[k]$ where $\tilde{x}[n]$ is a period-10 seq.

$$X[k] = \begin{cases} \sum_{n=0}^{4} W_{10}^{nk} & k = 0,1,2,\cdots,9 \\ 0 & \text{otherwise} \end{cases}$$

"10-point DFT"

Example

- Show:

$$X[k] = \sum_{n=0}^{4} W_{10}^{nk}$$
$$= e^{-j\frac{4\pi}{10}k} \frac{\sin(\frac{\pi}{2}k)}{\sin(\frac{\pi}{10}k)}$$

"10-point DFT"

DFT vs DTFT

- For finite sequences of length N:
  - The N-point DFT of x[n] is:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)nk} \qquad 0 \le k \le N-1$$

  - The DTFT of x[n] is:

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \qquad -\infty < \omega < \infty$$

What is similar?

DFT vs DTFT

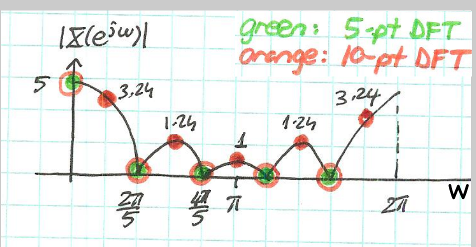
- The DFT are samples of the DTFT at N equally spaced frequencies

$$X[k] = X(e^{j\omega})|_{\omega = k\frac{2\pi}{N}} \qquad 0 \le k \le N-1$$

DFT vs DTFT

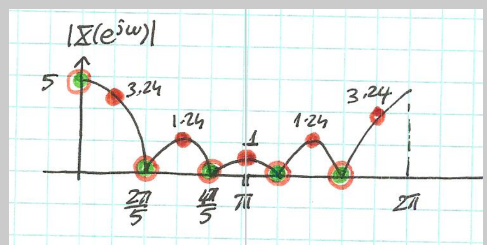
- Back to moving average example:

$$X(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n}$$
$$= e^{-j2\omega} \frac{\sin(\frac{5}{2}\omega)}{\sin(\frac{\omega}{2})}$$



FFTSHIFT

- Note that k=0 is w=0 frequency
- Use fftshift to shift the spectrum so w=0 in the middle.



DFT and Inverse DFT

- Both computed similarly.....let's play:

$$
\begin{aligned}
N \cdot x^*[n] &= N \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \right)^* \\
&= \sum_{k=0}^{N-1} X^*[k] W_N^{kn} \\
&= \mathcal{DFT}\left\{ X^*[k] \right\}.
\end{aligned}
$$

- Also....

$$N \cdot x^*[n] = N \left(\mathcal{DFT}^{-1}\left\{ X[k] \right\} \right)^*.$$

DFT and Inverse DFT

- So,

$$\mathcal{DFT}\left\{ X^*[k] \right\} = N \left(\mathcal{DFT}^{-1}\left\{ X[k] \right\} \right)^*$$

or,

–

$$\mathcal{DFT}^{-1}\left\{ X[k] \right\} = \frac{1}{N} \left(\mathcal{DFT}\left\{ X^*[k] \right\} \right)^*$$

- Implement IDFT by:
  - Take complex conjugate
  - Take DFT
  - Multiply by 1/N
  - Take complex conjugate !

Why useful?

DFT as Matrix Operator

DFT:

$$
\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix}
=
\begin{pmatrix}
W_N^{00} & \cdots & W_N^{0n} & \cdots & W_N^{0(N-1)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
W_N^{k0} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
W_N^{(N-1)0} & \cdots & W_N^{(N-1)n} & \cdots & W_N^{(N-1)(N-1)}
\end{pmatrix}
\begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}
$$

IDFT:

$$
\begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}
= \frac{1}{N}
\begin{pmatrix}
W_N^{-00} & \cdots & W_N^{-0k} & \cdots & W_N^{-0(N-1)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
W_N^{-n0} & \cdots & W_N^{-nk} & \cdots & W_N^{-n(N-1)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
W_N^{-(N-1)0} & \cdots & W_N^{-(N-1)k} & \cdots & W_N^{-(N-1)(N-1)}
\end{pmatrix}
\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix}
$$

straightforward implementation requires N^2 complex multiplies :-(

DFT as Matrix Operator

- Can write compactly as:

$$
\begin{aligned}
\mathbf{X} &= \mathbf{W}_N \ \mathbf{x} \\
\mathbf{x} &= \frac{1}{N} \ \mathbf{W}_N^* \ \mathbf{X}
\end{aligned}
$$

- So,

$$\mathbf{x} = \frac{1}{N}\mathbf{W}_N^*\mathbf{X} = \frac{1}{N}\mathbf{W}_N^*\mathbf{W}_N\mathbf{x} = \frac{1}{N}\left(N\mathcal{I}\right)\mathbf{x} = \mathbf{x}$$

WHY?

as expected.

Properties of DFT

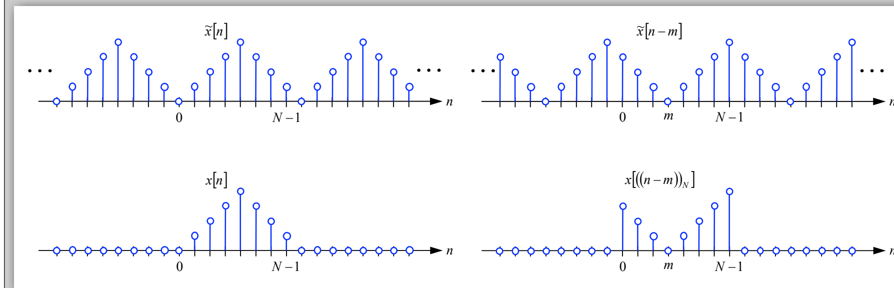
- Inherited from DFS (EE120/20) so no need to be proved
- Linearity

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1[k] + \alpha_2 X_2[k]$$

- Circular Time Shift

$$x[((n-m))_N] \leftrightarrow X[k]e^{-j(2\pi/N)km} = X[k]W_N^{km}$$

Circular shift



Properties of DFT

- Circular frequency shift

$$x[n]e^{j(2\pi/N)nl} = x[n]W_N^{-nl} \leftrightarrow X[((k-l))_N]$$

- Complex Conjugation

$$x^*[n] \leftrightarrow X^*[((-k))_N]$$

- Conjugate Symmetry for Real Signals

$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

Show....

Examples

- 4-point DFT
  - –Basis functions?
  - –Symmetry
- 5-point DFT
  - –Basis functions?
  - –Symmetry

Properties of DFT

- Parseval's Identity

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

- Proof (in matrix notation)

$$\mathbf{x}^*\mathbf{x} = \left(\frac{1}{N}\mathbf{W}_N^*\mathbf{X}\right)^*\left(\frac{1}{N}\mathbf{W}_N^*\mathbf{X}\right) = \frac{1}{N^2}\mathbf{X}^*\underbrace{\mathbf{W}_N\mathbf{W}_N^*}_{N\cdot\mathbf{I}}\mathbf{X} = \frac{1}{N}\mathbf{X}^*\mathbf{X}$$

Circular Convolution Sum

- Circular Convolution:

$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

for two signals of length N

- Note: Circular convolution is commutative

$$x_2[n] \circledN x_1[n] = x_1[n] \circledN x_2[n]$$

Properties of DFT

- Circular Convolution: Let x1[n], x2[n] be length N

$$x_1[n] \circledN x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$

Very useful!!! (for linear convolutions with DFT)

- Multiplication: Let x1[n], x2[n] be length N

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \circledN X_2[k]$$

Linear Convolution

- Next....
  - Using DFT, circular convolution is easy
  - But, **linear** convolution is useful, not circular
  - So, show how to perform linear convolution with circular convolution
  - Used DFT to do linear convolution