EECS 123 Digital Signal Processing — University of California, Berkeley: Fall 2007

Gastpar — November 7, 2007

Exam 2

Last name	First name	SID

- You have 1 hour and 45 minutes to complete this exam.
- The exam is closed-book and closed-notes; calculators, computing and communication devices are *not* permitted.
- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- However, four single-sided US letter pages of handwritten and *not photocopied* notes are allowed.
- Additionally, you receive Tables 2.1, 2.2, 2.3, 3.1, 3.2, 8.1, 8.2 from the class textbook.
- If we can't read it, we can't grade it.
- We can only give partial credit if you write out your derivations and reasoning in detail.
- You may use the back of the pages of the exam if you need more space.

*** Good Luck! ***

Problem	Points earned	out of
Problem 1		20
Problem 2		20
Problem 3		20
Problem 4		25
Problem 5		15
Total		100

Partial Credit.

Partial credit will be given only if there is sufficient information in your work. In general, a good way to show that you understand what's going on is for example to provide plots, sketches, and formulas for *intermediate* signals. That way, if you make an error somewhere along the way, we can trace it and evaluate whether or not you understood the basics of the problem.

Useful Formulae.

- For the continuous-time box function,

$$b(t) = \begin{cases} 1, & -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

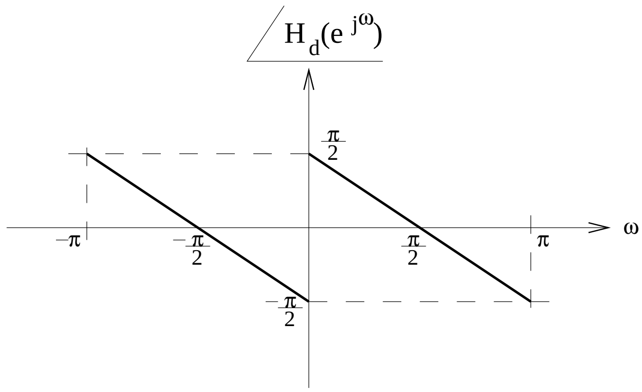
  the (continuous-time) Fourier transform is given by

$$B(j\Omega) = \frac{2\sin(\Omega T)}{\Omega}. \tag{2}$$

- $\tan(\pi/4) = 1$

Problem 1 *(Phase Properties.)* 20 Points

Given the phase characteristics of a generalized linear phase FIR filter $H_d(e^{j\omega})$ shown below, answer the following questions. Include brief explanations to get credit.



(a) (6 pts) Is this a symmetric (i.e., Type-I or Type-II) or an anti-symmetric (i.e., Type-III or Type-IV) filter? Why?

$\boxed{\text{Anti-symmetric}}$, because it has a $\pm\pi/2$ phase shift.

(b) (8 pts) Can the filter length be determined from the given information? If yes, what is the length? If not, why not?

$\boxed{\text{Yes}}$, the slope of the phase is -1. So, the delay is 1. Then $M = 2$ and $\boxed{N = 3}$

(c) (6 pts) The filter magnitude response has a DC gain of 1. True or False? Why?

$\boxed{\text{False}}$, an anti-symmetric even-order FIR filter must have a DC gain of zero.

Problem 2 *(Filter Design.)* 20 Points

(a) (12 pts) A continuous-time filter is given by $H_a(s) = \frac{2}{2+s}$. We want to use this as a prototype filter to design a discrete-time filter *via the bilinear transform* with a suppression of $1/\sqrt{2}$ at $\omega_c = \pi/2$. (Note that the filter has a gain of 1 at frequency zero.) Give $H(z)$ explicitly.

One way to solve this problem is follows. First determine the continuous frequency with attenuation $1/\sqrt{2}$.

$$|H_a(j\Omega_c)| = \left|\frac{2}{2 + j\Omega_c}\right| = \frac{1}{\sqrt{2}} \qquad \Rightarrow \Omega_c = 2$$

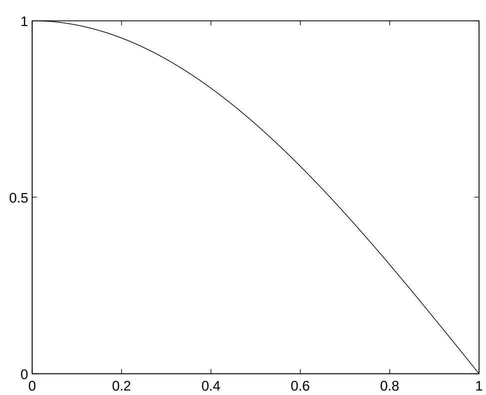
Since $\Omega_c = \frac{2}{T_d}\tan(\omega_c/2)$ this means that we want $T_d = 1$, and thus

$$H(z) = \frac{2}{2 + 2\frac{1-z^{-1}}{1+z^{-1}}} = \frac{1}{2}(1 + z^{-1})$$

Another approach is to solve for $H(z)$ in terms of T_d, and then to solve for the value of T_d such that $|H(e^{j\omega/2})| = 1/\sqrt{2}$.

(b) (8 pts) Sketch $|H(e^{j\omega})|$ for the filter designed in part (a) over the interval $0 \leq \omega \leq \pi$.

$$H(e^{j\omega}) = \frac{1}{2}e^{-j\omega/2}(e^{j\omega/2} + e^{-j\omega/2}) = e^{-j\omega/2}\cos(\omega/2)$$



Problem 3 *(Filter Design.)* 20 Points

(a) (5 pts) We want to approximate the lowpass filter in Figure 1 with the optimal minimax (Parks-McClellan) Type-I filter $h(n)$. Just like in class, the band $0 \leq \omega \leq \omega_p$ is the desired passband, and the band $\omega_s \leq \omega \leq \pi$ is the desired stopband. In Figure 1, provide a sketch the form of $H(e^{j\omega})$ when $h(n)$ has length 3. Recall that the amplitude of a Type-I filter has the form $A(e^{j\omega}) = \sum_{k=0}^{M/2} a_k \cos(k\omega)$. Assume that $\omega_s = \pi - \omega_p$.

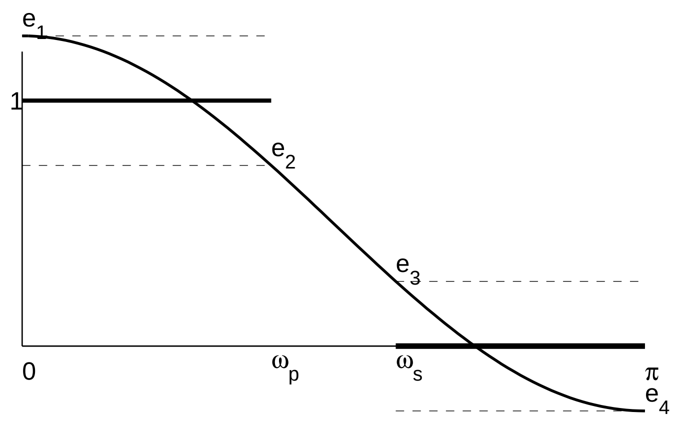


Figure 1:

(b) (10 pts) Determine the filter $h(n)$. *Hint:* If you find it easier, you may start by assuming that $\omega_p = \pi/3$ and thus, $\omega_s = 2\pi/3$.

Since the filter has length 3, we know that $A(w) = a_0 + a_1 cos(\omega)$. There must be alternations at the boundaries e_2 and e_3. Also, because $cos(\omega)$ is strictly decreasing on the interval $[0, \pi]$ it is clear that an alternation point must occur at e_1 or e_4. Given the symmetry of the problem and the fact that $A(w)$ is just a shifted and stretched cosine, we see that the error at e_1 and e_4 must be the same, that is they are both alternation points. Using this reasoning, we have $a_0 = 1/2$. Equating the error at e_1 and e_2 gives

$$a_0 + a_1 \cos(0) - 1 = 1 - a_0 - a_1 \cos(\omega_p) \quad \Rightarrow \quad a_1 = \frac{1}{1 + \cos(\omega_p)}.$$

And so the filter is given by

$$h = [a_1/2, a_0, a_1/2]$$

Note that if $w_p = \pi/3$, then $h = [1/3, 1/2, 1/3]$.

(c) (5 pts) What is the largest value of ω_p such that the maximum error is $\delta \leq 1/6$?

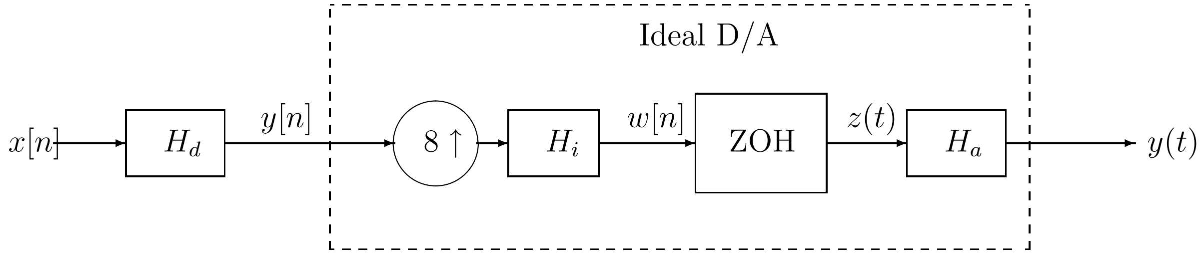
The maximum error is

$$\delta = a_0 + a_1 - 1 = \frac{1}{1 + \cos(\omega_p)} - \frac{1}{2}$$

Solving this leads to

$$\delta \leq 1/6 \quad \Rightarrow \quad \cos(\omega_p) \geq 1/2 \quad \Rightarrow \quad \omega_p \leq \pi/3$$

Problem 4 *(Multirate System.)* 25 Points



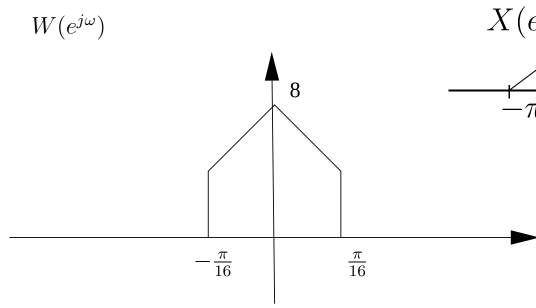
$$H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{\pi}{2} \\ 0, & \text{else} \end{cases} \qquad H_i(e^{j\omega}) = \begin{cases} 8, & |\omega| \le \frac{\pi}{8} \\ 0, & \text{else} \end{cases}$$

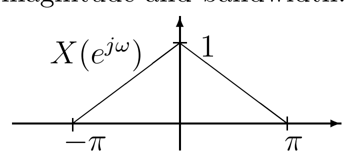
The ZOH operates at interval T but produces pulses of width $\frac{T}{4}$, i.e.

$$g(t) = \begin{cases} 1, & 0 \le t \le \frac{T}{4} \\ 0, & \text{else,} \end{cases}$$

and the output is $z(t) = \sum_{n=-\infty}^{\infty} w[n]g(t - nT)$.

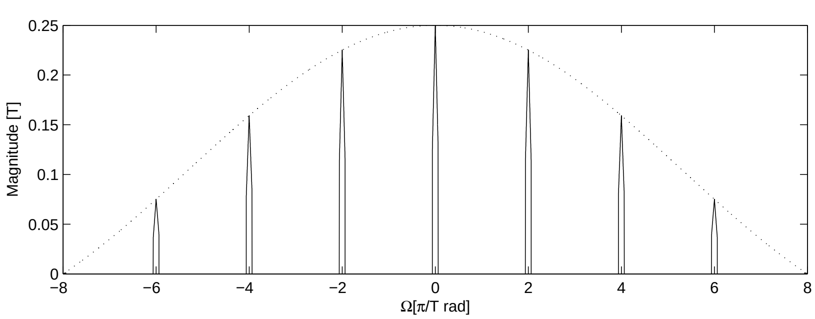
(a) (9 pts) For $X(e^{j\omega})$ pictured, sketch $W(e^{j\omega})$. Label the magnitude and bandwidth.



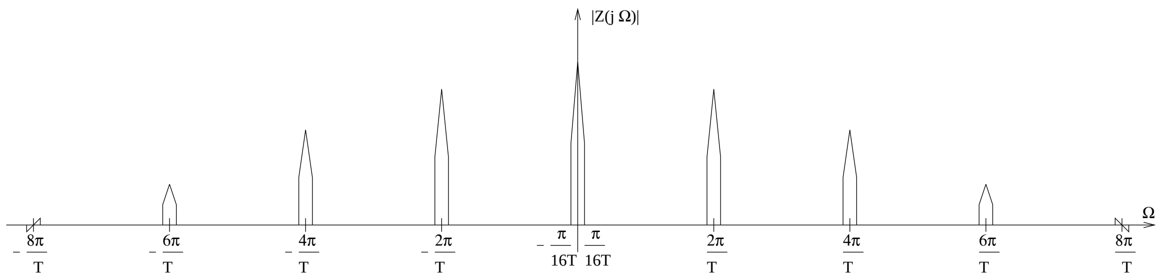


(b) (7 pts) For the same $X(e^{j\omega})$, sketch $|Z(j\Omega)|$ for $\Omega = [\frac{-8\pi}{T}, \frac{8\pi}{T}]$.

$$\text{rect}\left(\frac{t}{T/4}\right) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{T}{4}\text{sinc}\left(\Omega\frac{T}{8}\right) = 2\frac{\sin(\Omega T/8)}{\Omega}$$



See the next page for a more accurate depiction of the labels

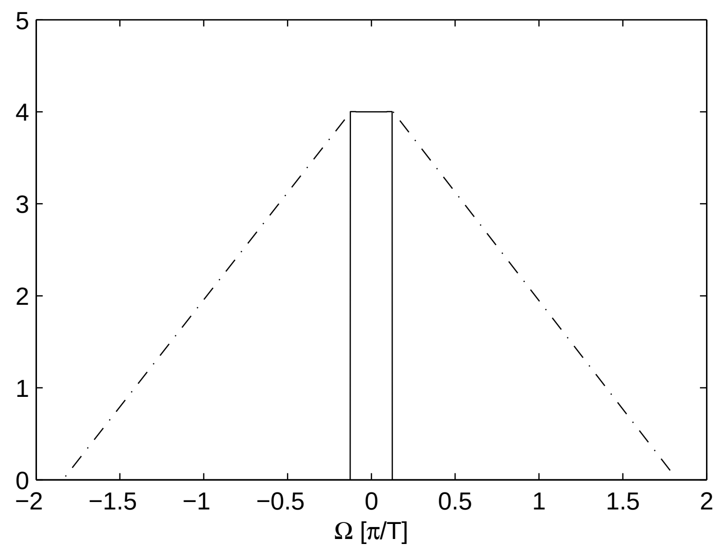


|Z(j Ω)|
Ω
$-\frac{8\pi}{T}$
$-\frac{6\pi}{T}$
$-\frac{4\pi}{T}$
$-\frac{2\pi}{T}$
$-\frac{\pi}{16T}$
$\frac{\pi}{16T}$
$\frac{2\pi}{T}$
$\frac{4\pi}{T}$
$\frac{6\pi}{T}$
$\frac{8\pi}{T}$

(c) (7 pts) For this part, we are interested in making the dashed block (with input $y[n]$ and output $y(t)$) an ideal D/A converter for arbitrary $y[n]$, i.e. ignore the effects of H_d.

What is the "cheapest"[1] filter $H_a(j\Omega)$ such that between $y[n]$ and $y(t)$ we have an ideal D/A. Sketch $|H_a(j\Omega)|$ and specify its value where necessary.

We start by noting that the signal $y[n]$ used to arrive at the figure in Part (b) only occupied half of the frequency band. In order to obtain a general ideal D-to-A converter, we must allow for input signals $y[n]$ that occupy the entire (digital) spectrum, and thus, the components of the corresponding signal $z(t)$ will have a width of $2 \cdot \frac{\pi}{8T}$ (rather than only $2 \cdot \frac{\pi}{16T}$, as in the figure in Part (b)). However, the crucial point is that the spectrum $Z(j\Omega)$ is *zero* for all frequencies $\frac{\pi}{8T} < |\Omega| < \frac{15\pi}{8T}$, and we can exploit this fact in the design of the filter $H_a(j\Omega)$. Specifically, that filter must be the inverse of $G(j\Omega)$ inside the band $0 \leq |\Omega| \leq \frac{\pi}{8T}$, but can be *anything* in the band $\frac{\pi}{8T} < |\Omega| < \frac{15\pi}{8T}$. We can exploit this degree of freedom and use a "cheaper" filter, such as the one drawn in the figure below.



(d) (2 pts) Explain (in words) the advantages and disadvantages of this D/A converter design over a direct implementation (as we have discussed it in class).

The advantage is that we no longer need an ideal continuous-time filter in order to implement an ideal digital-to-analog conversion. Rather, we can simply use a relaxed filter such as the one found in Part (c). That filter is cheaper to implement. The disadvantage is that we need more digital logic. (However, for the past few decades, digital has become cheaper and faster according to Moore's law (check it out on Wikipedia if you have never heard about it...), and so, this has not been considered a real issue.) Oversampling is a trick used in many digital-analog (and also analog-digital, as we will see very soon) conversion systems.

[1] the filter having the largest transition band (as we have seen in class, the smaller (i.e., steeper) the transition band, the more filter coefficients are necessary)

Problem 5 *(Filter Bank.)* 15 Points

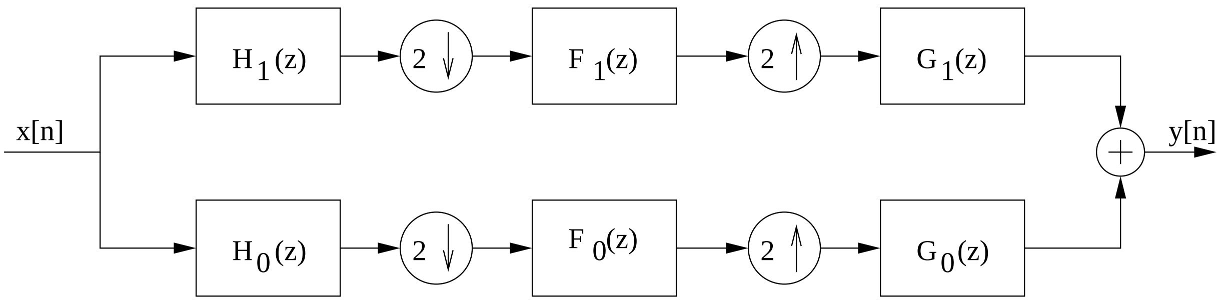


Figure 2: A two-channel filter bank.

For the filter bank in Figure 2, find the conditions for perfect reconstruction, i.e., the conditions on the filters $F_i(z), G_i(z), H_i(z)$ (for $i = 0, 1$) such that $y[n] = x[n]$.

We may replace $F_i(z)$ with the filter $F_i(z^2)$ which occurs either before the downsampling or after the upsampling. Either way we get

$$G_0(z)F_0(z^2)H_0(z) + G_1(z)F_1(z^2)H_1(z) = 2$$

and

$$G_0(z)F_0(z^2)H_0) - z) + G_1(z)F_1(z^2)H_1(-z) = 0$$

as the conditions for perfect reconstruction.