Midterm

EE 123

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Spring 2008

The midterm has 3 (THREE) Questions. There are 12 (TWELVE) pages total.

Problem	Max	Points
1	35	
2	30	
3	35	
Total	100	

- DO NOT open the exam until instructed to do so.
- This is a closed book exam.
- You are allowed ONE side of a 8.5x11 inch sheet.
- You have 120 minutes to finish this exam.
- Box your final answers.
- Partial marks will not be awarded to answers that have no proper reasoning.
- Answers arrived at with the aid of programmable calculators, which do not show insight into the problem will not fetch any credit.
- **Show your work to receive credit.**
- Remember to write your name and SID on the top right corner of every sheet of paper.
- You may use the empty pages to do your work.

NAME: Solution

SID: ____________

- **Problem 1**

I.) For each of the following LTI systems, determine if the system is causal and/or BIBO stable. Indicate "**Y**" for Yes, "**N**" for No, and "**X**" for "cannot be determined due to insufficient information." If the number of poles in the system's transfer function can be determined, write the answer. If not, write an "**X**" as the answer. (*3 + 3 points*)

a.) Let $x[n]$ and $y[n]$ denote the system input and output, respectively of a discrete time LTI system. Let $x[n] = (\cos\omega_0) \cdot y[n] + (\sin\omega_0) \cdot y[n-1]$, where $0 < \omega_0 < \frac{\pi}{2}$ is a constant. Fill in the following blanks for this system,

X Stable Y Causal 1 Number of poles

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{\cos\omega_0 + \sin\omega_0 \cdot z^{-1}}$$; note that $\cos\omega_0$ and $\sin\omega_0$ are constants

- $H(z)$ has one pole at $z = -\tan\omega_0$
- for $|\omega_0| > \frac{\pi}{4}$, system is unstable and for $|\omega_0| < \frac{\pi}{4}$, system is stable.
- System is causal since $y[n]$ depends on $x[n]$ and $y[n-1]$.

b.) Let $h[n]$ be the impulse response of the LTI system, with $\left(\frac{1}{2}\right)^n < h[n] < \left(\frac{5}{6}\right)^n$, $n \geq 0$; and $h[n] = 0, \quad n < 0$. Fill in the following blanks for this system,

Y Stable Y Causal X Number of poles

- Clearly system with impulse response $h[n]$ is causal since $|h[n]| = 0$ for $n < 0$.

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n < \sum_{n=0}^{\infty}|h[n]| < \sum_{n=0}^{\infty}\left(\frac{5}{6}\right)^n \quad \text{(By given condition)}$$

$$\Rightarrow \frac{1}{1-\frac{1}{2}} < \sum_{n=0}^{\infty}|h[n]| < \frac{1}{1-\frac{5}{6}} \Rightarrow \sum_{n=0}^{\infty}|h[n]| \text{ is finite}$$

$\Rightarrow$ BIBO stable.

- $h_1[n] = \left(\frac{2}{3}\right)^n u[n]$ and $h_2[n] = \frac{1}{2}\left\{\left(\frac{1}{2}\right)^n u[n] + \left(\frac{5}{6}\right)^n u[n]\right\}$ satisfy the criteria of $h[n]$, and have 1 and 2 poles respectively.

II.) Let N be a positive integer. Let $y[n] = x\left[\frac{n}{N}\right]$, when n is divisible by N, and $y[n] = 0$, otherwise. Find a closed form expression for $Y(z)$ in terms of $X(z)$.[1] *(5 points)*

$Y(z) =$ $X(z^N)$

$$Y(z) = \sum_{n=\infty}^{\infty} y[n] \cdot z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} x[m] \cdot z^{-mN}$$

$m = \frac{n}{N}$

$$= \sum_{m=-\infty}^{\infty} x[m] \cdot (z^N)^{-m}$$

$$= X(z^N)$$

III.) Let a discrete time LTI system be given by the following difference equation:

$$y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n].$$

For $x[n] = (-1)^n$, find $y[n]$ for $n = -20, 0, 19$. Show your work to get credit. *(6 points)*

$y[-20] =$ 4 $y[0] =$ 4 $y[19] =$ -4

We will use the eigenfunction property of LTI systems.

$$\underset{e^{j\pi n}}{\overset{\|}{(-1)^n}} \rightarrow \boxed{h[n]} \rightarrow H(e^{j\pi}) \cdot (-1)^n$$

$$H(e^{j\omega}) = \frac{1}{1 + e^{-j\omega} + \frac{1}{4}e^{-j2\omega}}$$

$$\Rightarrow H(e^{j\pi}) = \frac{1}{1 - 1 + \frac{1}{4}} = 4$$

[1] A closed-form expression contains no summation, integrals, etc.

IV.) The pole-zero plot for a transfer function $H(z)$ for a discrete-time LTI system is shown in Figure 1. *(2 + 2 + 2 points)*

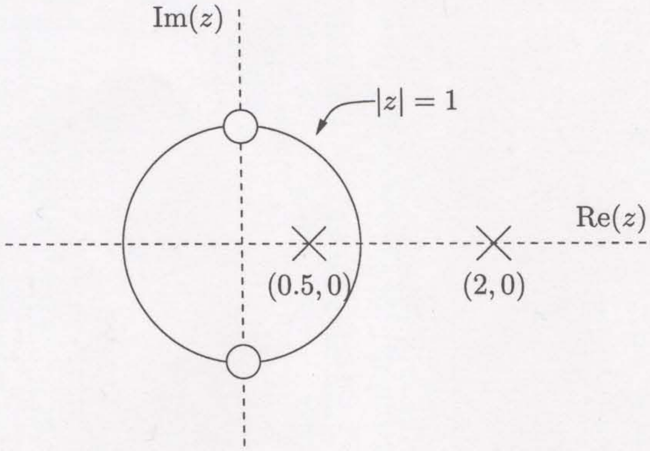


Figure 1: The pole-zero plot of $H(z)$.

a.) Write the system transfer function $H(z)$ that is associated with the pole-zero plot in Figure 1, to within a scaling constant.

$H(z) =$ $\dfrac{k \cdot (z^2+1)}{(z^2 - \frac{5}{2}z + 1)}$

$$H(z) = K \cdot \frac{(z-j)(z+j)}{(z-\frac{1}{2})(z-2)} = \frac{k \cdot (z^2+1)}{(z^2 - \frac{5}{2}z + 1)}$$

b.) Can a system with this pole-zero plot be *both* stable and causal? Write "**X**", if the pole-zero plot information is insufficient to determine the answer. Explain to get credit.

No Yes/No/**X**

For stability $|z| = 1$ should be in ROC.
ROC cannot have any poles.
$\Rightarrow$ ROC should be $\frac{1}{2} < |z| < 2$, for stability
$\Rightarrow$ system is not causal.

c.) If the system is stable, can the impulse response $h[n]$ be single-sided? Write "**X**", if the pole-zero plot information is insufficient to determine the answer. Explain to get credit.

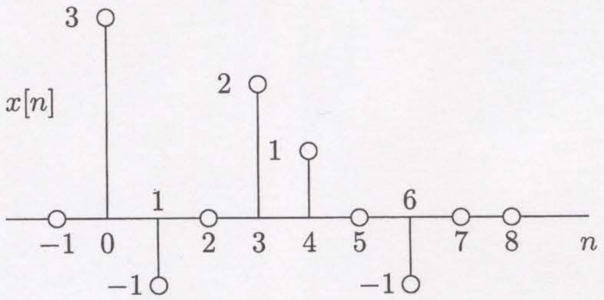
No YES/NO/**X**

(Same argument as (b)).

ROC is $\frac{1}{2} < |z| < 2$.

$\Rightarrow h[n]$ is double-sided.

V.) Let $X(e^{j\omega})$ be the DTFT of $x[n]$ shown below. Define $Y[k] = X(e^{j\omega})\,|_{\omega=\frac{2\pi k}{4}}$ where $0 \leq k \leq 3$. (*6 points*)



Sketch the signal $y[n]$ which is the 4-point inverse DFT of $\{Y[k]\}_{k=0}^{3}$.

Due to time-aliasing,

$$y[n] = \sum_{l=-\infty}^{\infty} x[n-4l].$$

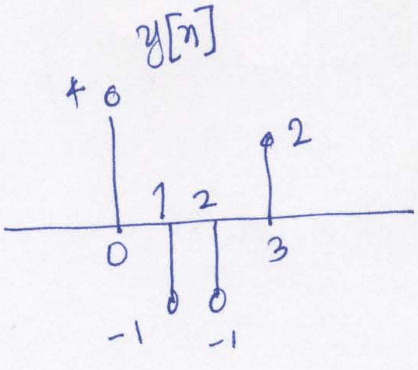
$\therefore$

$y[0] = x[0] + x[4] = 4$

$y[1] = x[1] + x[5] = -1$

$y[2] = x[2] + x[6] = -1$

$y[3] = x[3] + x[7] = 2$



VI.) Let a 64-point FFT be available as a module (black box). You are given a sequence $\{x[n]\}_{n=0}^{4900}$ and a 15-point filter $\{h[n]\}_{n=0}^{14}$. You have to compute the linear convolution $y[n] = x[n] * h[n]$ using the DFT method i.e., by using the equation $DFT^{-1}[DFT(x[n]) \cdot DFT(h[n])]$.

Consider the *overlap-save* method. Find the number of DFT's and inverse DFT's needed for the given task. *Assume that the 64-point DFT of h[n] is pre-computed and available.* (*6 points*)

99 # of DFTs 99. # of IDFTs

Length of $x[n]$: $l_x = 4901$

- Pad 14 zeros befor $x[n]$; ie from $n=-14$ to $n=-1$.
- In each ~~FFT~~ $DFT^{-1}[DFT(x[n]) \cdot DFT(h[n])]$ operation $64-15+1 = 50$ samples of $x[n] * h[n]$ are obtained. $(N-M+1)$
- Total number of DFTs needed $\left\lceil \frac{4901}{50} \right\rceil = 99$.
- Number of IDFTs is same since $H[k]$ are pre-computed

- **Problem 2**

When John Smith plugged in his new digital TV, he was dismayed to see ghosts on the screen. Some measurements revealed that this problem could be modeled by his digital video signal being filtered by an LTI system with unit pulse response

$$h[n] = \delta[n] - 0.1\delta[n-64].$$

To correct the problem, he wishes to process his signal by an inverse filter with unit pulse response $g[n]$, so that the effect of $h[n]$ is canceled. To determine $g[n]$ he computes the N-point DFT $\{H[k]\}_{k=0}^{N-1}$, with $N = 128$, of the sequence $h[n]$ and then defines $g[n]$ as the inverse DFT of $G[k] = 1/H[k]$, for $k = 0, 1, \ldots, N-1$.

(a) Find the transfer function of the exact inverse of the filter $h[n]$. (*5 points*)

$$H(z) = 1 - (0.1)z^{-64}$$

$\therefore$ exact inverse of $H(z)$, say $F(z)$, is

$$\boxed{F(z) = \frac{1}{1 - 0.1 \cdot z^{-64}}}$$

Note that $f[n]$ has an infinite length impulse response.

(b) Determine the DFT coefficients $\{G[k]\}_{k=0}^{N-1}$. (*5 points*)

$$H(e^{j\omega}) = 1 - 0.1\, e^{-j64\omega}$$

$$\Rightarrow H[k] = 1 - 0.1\, e^{-j64 \cdot \frac{2\pi k}{128}}$$

128pt

$$= 1 - 0.1\, e^{-j\pi k} \; ; \; 0 \le k \le 127$$

$$\Rightarrow \boxed{G[k] = \frac{1}{1 - 0.1\, e^{-j\pi k}} \; ; \quad 0 \le k \le 127}$$

(c) Determine $g[n]$, the 128-point inverse of $\{G[k]\}_{k=0}^{N-1}$. (*10 points*)

$$G[k] = \frac{1}{1 - 0.1\, e^{-j\pi k}}$$

$$= \frac{1 + 0.1\, e^{-j\pi k}}{1 - 0.01 \cdot \underbrace{e^{-j2\pi k}}_{=1}}$$

$$= \frac{100}{99}\left(1 + 0.1\, e^{-j\pi k}\right); \quad 0 \le k \le 127$$

$$\Rightarrow \boxed{g[n] = \frac{100}{99}\left(\delta[n] + 0.1 \cdot \delta[n-64]\right);}$$

Note:- $e^{-j\pi k} = e^{-j\frac{2\pi}{128}\cdot 64k} \longleftrightarrow \delta[n-64]$ (see part (b)).

(d) Detemine the linear convolution, $g[n] * h[n]$. Is the system with impulse response $g[n]$ the inverse of the one with impulse response $h[n]$? Why or why not? Explain to get credit. (*10 points*)

$$G(z) = \frac{100}{99}\left(1 + 0.1 \cdot z^{-64}\right).$$

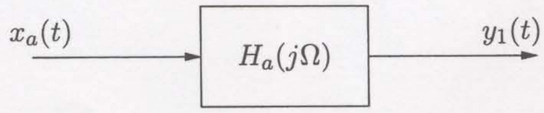
$$H(z) = \left(1 - 0.1 \cdot z^{-64}\right)$$

$$\Rightarrow G(z)\cdot H(z) = \frac{100}{99}\left(1 - 0.01 \cdot z^{-128}\right)$$

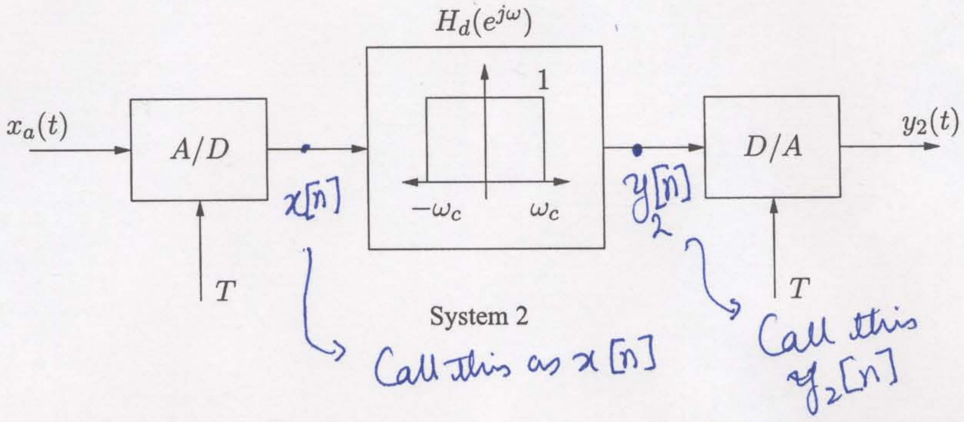
$$\Rightarrow \boxed{g[n] * h[n] = \frac{100}{99}\left(\delta[n] - 0.01\, \delta[n-128]\right)} \neq \delta[n].$$

$h[n] * g[n] \neq \delta[n] \Rightarrow g[n]$ is not the exact inverse.

- **Problem 3**



System 1



System 2

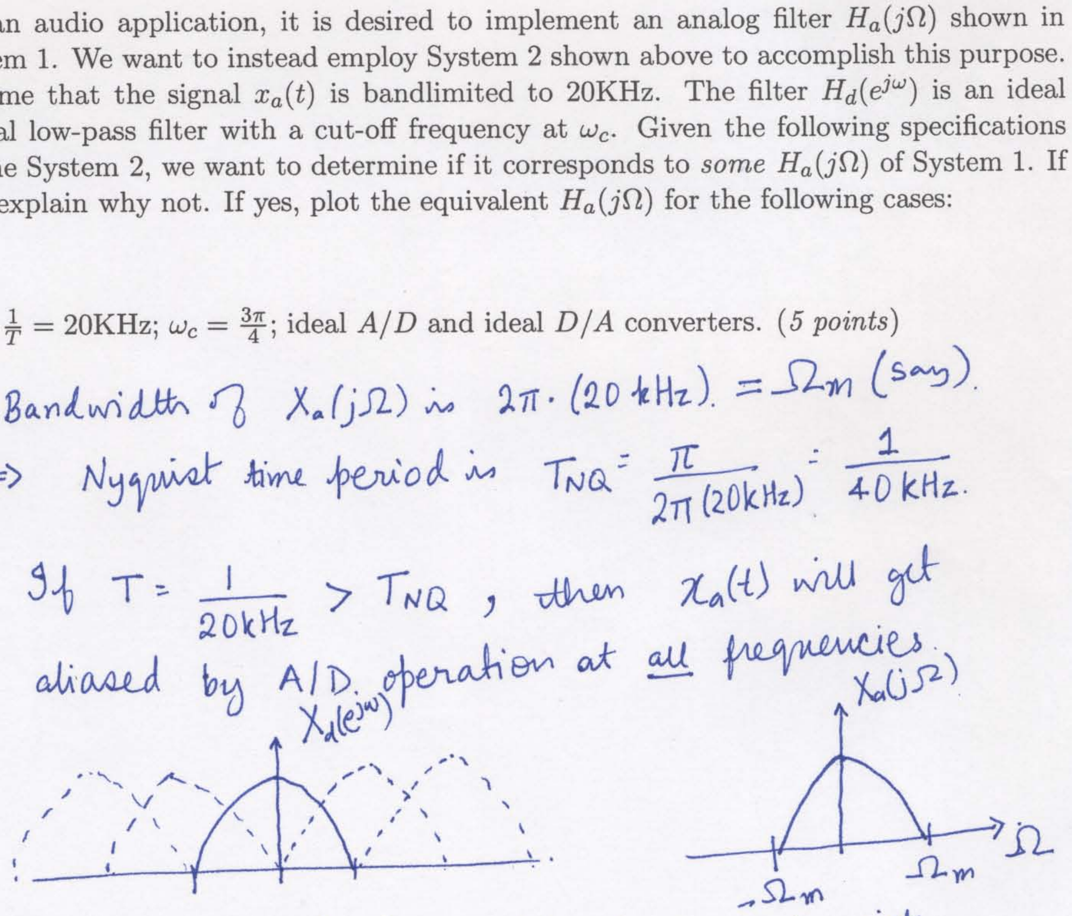
I. For an audio application, it is desired to implement an analog filter $H_a(j\Omega)$ shown in System 1. We want to instead employ System 2 shown above to accomplish this purpose. Assume that the signal $x_a(t)$ is bandlimited to 20KHz. The filter $H_d(e^{j\omega})$ is an ideal digital low-pass filter with a cut-off frequency at ω_c. Given the following specifications on the System 2, we want to determine if it corresponds to *some* $H_a(j\Omega)$ of System 1. If not, explain why not. If yes, plot the equivalent $H_a(j\Omega)$ for the following cases:

(i) $\frac{1}{T} = 20\text{KHz}$; $\omega_c = \frac{3\pi}{4}$; ideal A/D and ideal D/A converters. (*5 points*)

Bandwidth of $X_a(j\Omega)$ is $2\pi \cdot (20\,\text{kHz}) = \Omega_m$ (say).

$\Rightarrow$ Nyquist time period is $T_{NQ} = \frac{\pi}{2\pi(20\text{kHz})} = \frac{1}{40\,\text{kHz}}$.

If $T = \frac{1}{20\text{kHz}} > T_{NQ}$, then $x_a(t)$ will get aliased by A/D operation at all frequencies.

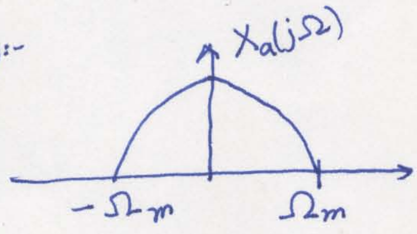


$\Rightarrow$ Equivalent $H_a(j\Omega)$ will not exist

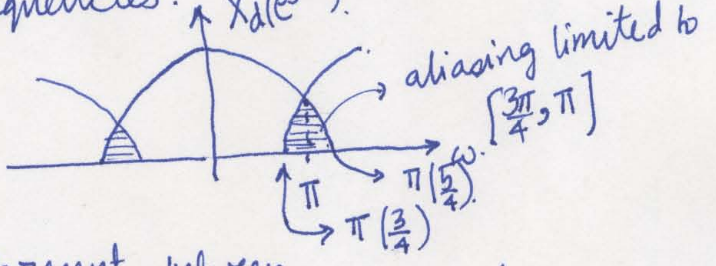
(ii) $\frac{1}{T} = 32\text{KHz}$; $\omega_c = \frac{3\pi}{4}$; ideal A/D and ideal D/A converters. (*7 points*)

$T = \frac{1}{32\,kHz} > T_{NQ}$. However, aliasing is not present at all frequencies.

E.g.:-

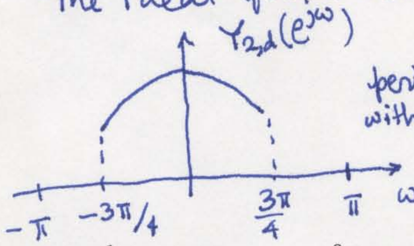


$\longrightarrow$

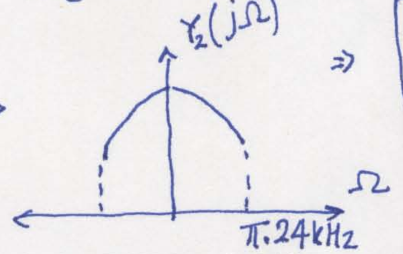


Observe that aliasing is present between $0.75\pi \le \omega \le \pi$. and $-0.75\pi \ge \omega \ge -\pi$.

The ideal filter with cut-off ω_c removes this aliased portion.



$\Rightarrow$

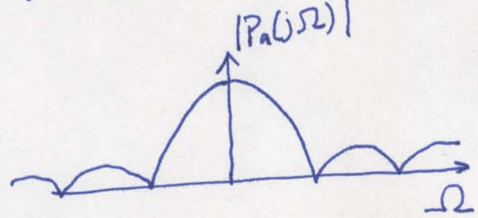


$\Rightarrow$ $\boxed{H_a(j\Omega) \text{ is an ideal lowpass filter with cut off at } 12kHz}$

(iii) $\frac{1}{T} = 40\text{KHz}$; $\omega_c = \frac{3\pi}{4}$; ideal A/D converter and a ZOH based D/A converter. (*8 points*)

No aliasing! $\quad y_2(t) = \sum_{n=-\infty}^{\infty} y_2[n] \cdot p_a(t-nT)$; $p_a(t) = u(t) - u(t-T)$.

$$Y_2(j\Omega) = \underbrace{Y_{2,d}(e^{j\Omega T})}_{\text{periodic and non-zero } \cancel{\text{outside}} \text{ for } \frac{|\Omega|}{2\pi} > (20kHz)} \cdot \underbrace{P_a(j\Omega)}_{\text{Non-zero for } \frac{|\Omega|}{2\pi} > 20kHz}.$$

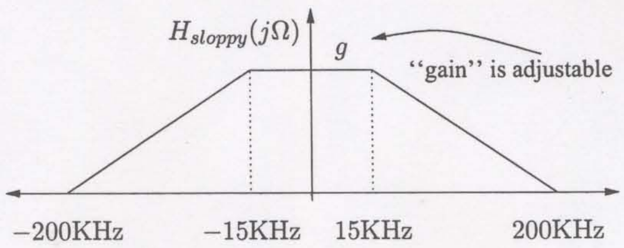


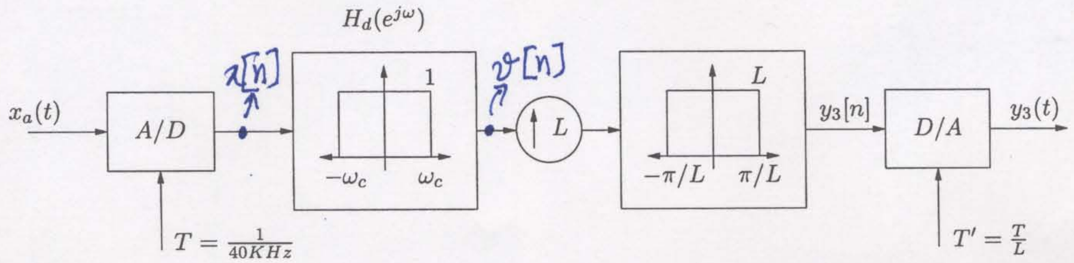
$\Rightarrow$ There exists output frequencies in $y_2(t)$ with non-zero energy corresponding to zero-energy ~~ind~~ input frequencies.

$\Rightarrow$ System with ZOH D/A is not even LTI.

$\Rightarrow$ equivalent filter is not possible. $(H_a(j\Omega))$

II. Now we are given the requirement that $H_a(j\Omega)$ in System 1 needs to be an ideal analog low-pass filter with cutoff frequency at 15Khz. Unfortunately, we have only a "sloppy" low-pass filter shown below (note that the gain g is adjustable):





$y_3(t) = \sum_{n=-\infty}^{\infty} y_3[n]h_{sloppy}(t - nT')$

System 3

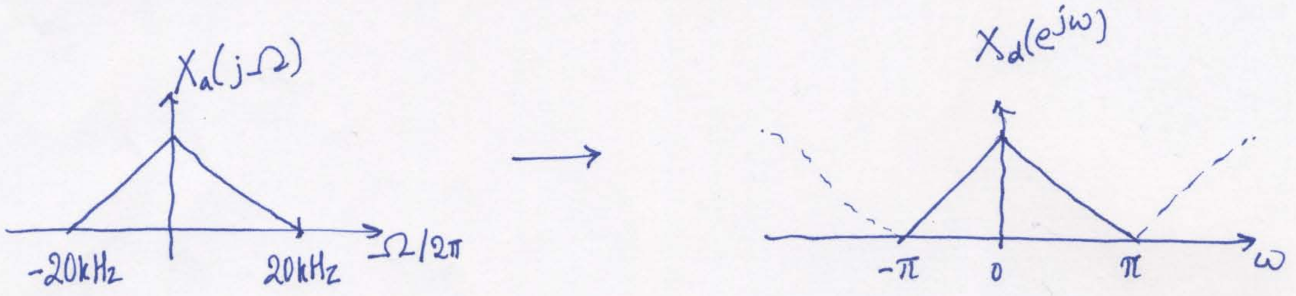
Can we still accomplish our objective $H_a(j\Omega)$ using sloppy filter $H_{sloppy}(j\Omega)$ in System 3? (See the figure above). If your answer is no, explain why not. If your answer is yes, what value of ω_c and what *minimal* value of L will achieve your design target? The output of the D/A converter is $y_3(t)$, where $y_3(t) = \sum_{n=-\infty}^{\infty} y_3[n]h_{sloppy}(t - nT')$, and $T' = \frac{T}{L}$ (see System 3 in the figure on the next page). Note that $H_d(e^{j\omega})$ is an ideal digital low-pass filter with cut-off frequency at ω_c, and L is a positive integer. (*15 points*)

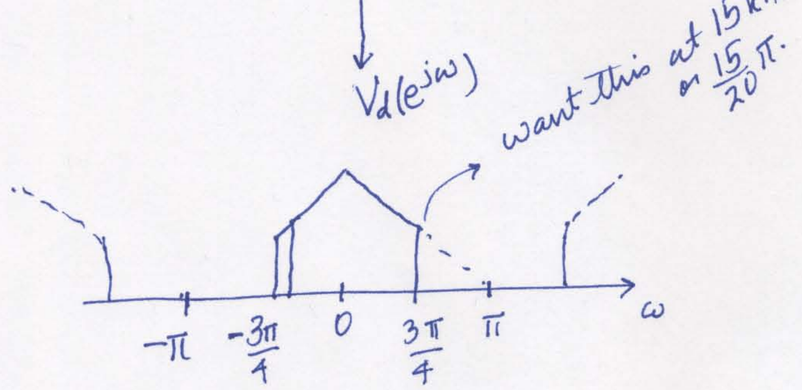
$T = T_{NQ} \Rightarrow$ No aliasing.

Note that $y_3(t) = \left(\sum_n y_3[n] \cdot \delta(t - nT')\right) * h(t)$.

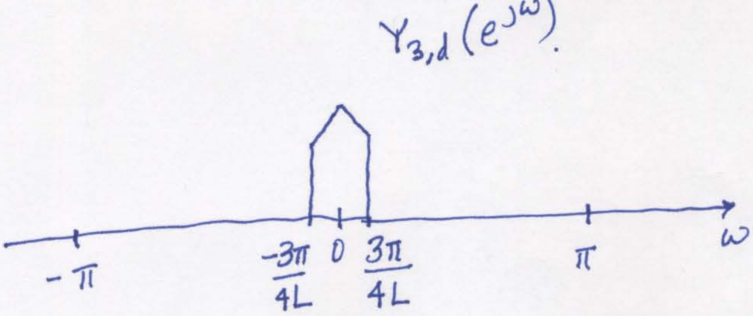
We will use an example $X_a(j\Omega)$ to gain insight into the problem. $x[n]$ and $v[n]$ are labeled above.

(See Next page)





$$\boxed{\text{For ideal LPF, } \omega_c = \frac{3\pi}{4}}$$



$\sum y_3[n] \cdot \delta(t-nT')$.

From figure, if

$L(40\text{kHz}) - 15\text{kHz} > 200\text{kHz}$,

then sloppy filter will do the desired job.

$\Rightarrow L > \frac{215}{40}$

or $L \geq 6$ (L is integer)

$\therefore \boxed{L_{min} = 6}$ is the answer.