EECS 123 Digital Signal Processing
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The Power Density Spectrum

Preliminary version — please report errors, typos, and suggestions for improvements

This handout is designed to be read **without any of the footnotes.** On a second reading, *if you are interested,* you may read some or all of the footnotes. If you are even more interested, you can come to office hours. Another resource is Appendix A of the course textbook [1].

1 The Two Main Properties

1.1 The Power Density Spectrum

We start by remembering that the power of a signal $x[n]$ is defined as

$$\sigma_x^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2. \quad (1)$$

While this formula may look a little complicated, it is really just the average of the squares of the signal values! (You have seen this formula in EECS 120.)

A random signal $x[n]$ can be characterized[1] by its *power density spectrum*

$$P_{xx}(\omega), \quad (2)$$

which is a 2π-periodic function saying, typically, how much power the signal $x[n]$ has at frequency ω, for $-\pi < \omega \leq \pi$. Because power is never negative, we must have that $P_{xx}(\omega) \geq 0$. An example of a power spectrum for a signal $x[n]$ is given in Figure 1.

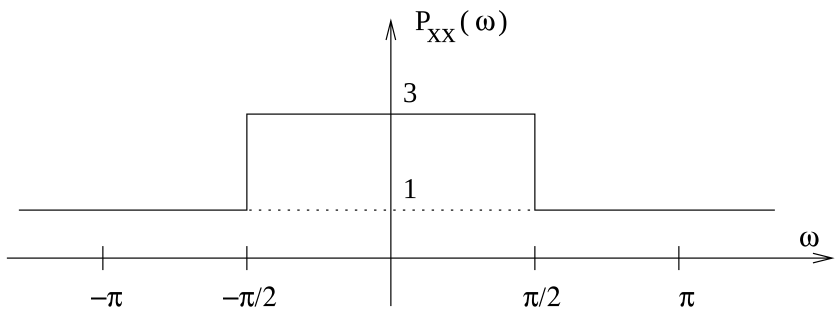

Figure 1: An example of a power spectrum: This signal has "more power at low frequencies."

Property 1. *The power σ_x^2 of a random signal $x[n]$ can be found from its power density spectrum $P_{xx}(\omega)$ simply by integrating:*

$$\sigma_x^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_{xx}(\omega) d\omega. \quad (3)$$

[1]More precisely, a class of random signals called *stationary and ergodic* can be characterized in this fashion.

Equivalently, the power of a random signal can also[2] be found as in Equation (1).

For the signal $x[n]$ whose power spectrum is given in Figure 1, we find

$$\begin{aligned} \sigma_x^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\omega)d\omega & (4) \\ &= \frac{1}{2\pi}\left(\int_{-\pi}^{-\pi/2} 1d\omega + \int_{-\pi/2}^{\pi/2} 3d\omega + \int_{\pi/2}^{\pi} 1d\omega\right) & (5) \\ &= \frac{1}{2\pi}\left(\frac{\pi}{2} + 3\pi + \frac{\pi}{2}\right) & (6) \\ &= 2. & (7) \end{aligned}$$

1.2 White Noise

A very important random signal is called *white noise,* for us denoted by $e[n]$. Its special property is that the power spectrum is *flat:*

$$P_{ee}(\omega) = \sigma_e^2, \text{ for all } \omega. \quad (8)$$

It is easy to verify that the power of the white noise $e[n]$ is then simply

$$\sigma_e^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{ee}(\omega)d\omega = \sigma_e^2. \quad (9)$$

1.3 The Filtering Property

The most important (and useful) property of the power density spectrum concerns LTI systems (i.e., filters):

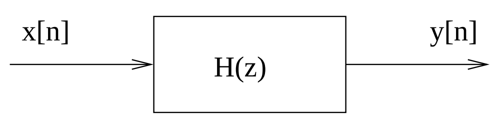

Figure 2: Passing a signal $x[n]$ through an LTI system $H(z)$, as we have done many times. Now, $x[n]$ is a random signal.

Property 2. *If we pass a random signal $x[n]$ through an LTI system with transfer function $H(z)$, the the output is also a random signal, call it $y[n]$, and its power density spectrum is given by*

$$P_{yy}(\omega) = |H(e^{j\omega})|^2 P_{xx}(\omega). \quad (10)$$

Let's again look at the signal $x[n]$ with power density spectrum as given in Figure 1. An example of filtering this signal is shown in Figure 3.

[2]This is a bit surprising: since the signal is random, really, the power as in Equation (1) is *also* random. However, for the important class of *ergodic* signals, the limit converges to a constant.

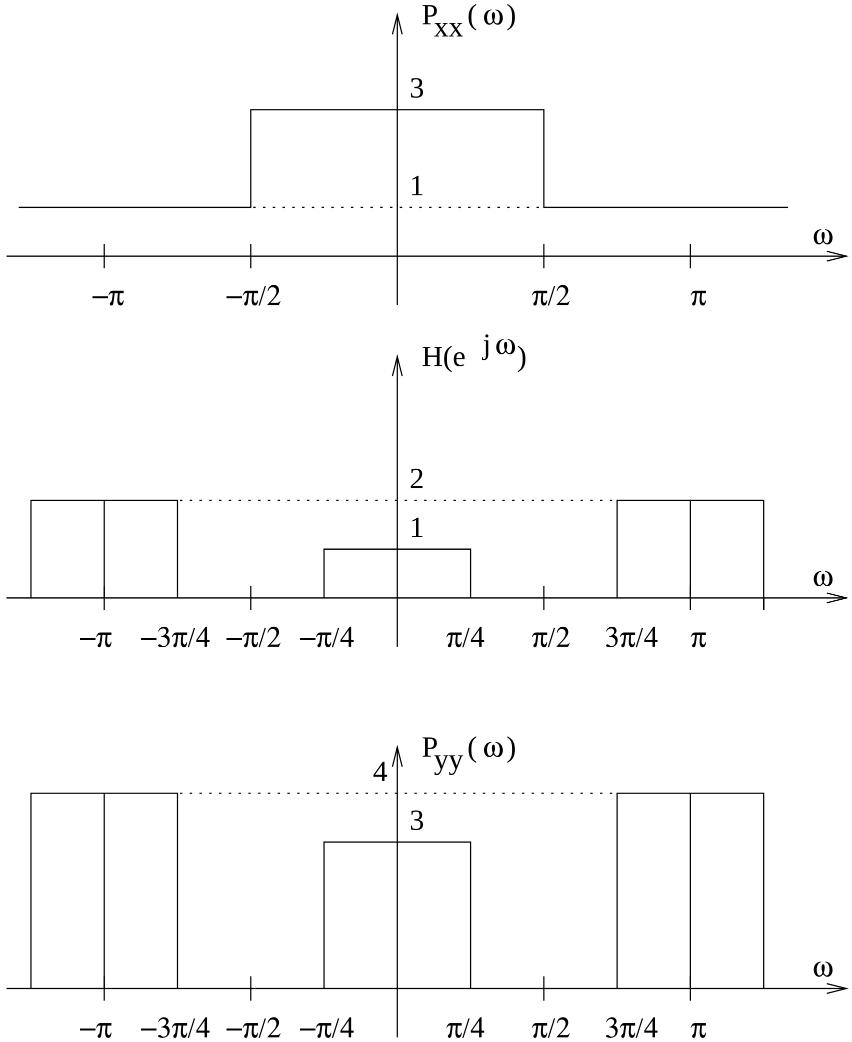

Figure 3: The signal $x[n]$ is passed through the filter $H(e^{j\omega})$, resulting in a signal $y[n]$ with different power density spectrum.

2 Downsampling and Upsampling

2.1 Downsampling

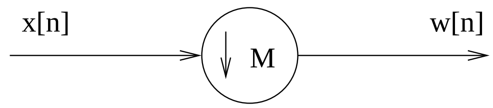

Figure 4: Downsampling by a factor of M.

Property 3. *Downsampling does not change the power:*

$$\sigma_w^2 = \sigma_x^2. \tag{11}$$

Property 4. *Let $x[n]$ be* **white noise.** *Then, $w[n]$ is also white noise.*

Note: If the input $x[n]$ is *not* white noise, then it is a bit more tricky to find the power density spectrum of $w[n]$. This is beyond the scope of this class.

2.2 Upsampling

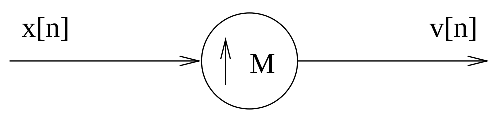

Figure 5: Upsampling by a factor of M.

Property 5. *Upsampling reduces the power by a factor of M :*

$$\sigma_v^2 = \frac{\sigma_x^2}{M}. \tag{12}$$

Property 6. *Let $x[n]$ be* **white noise.** *Then, $v[n]$ is also white noise.*

Note: If the input $x[n]$ is *not* white noise, then it is a bit more tricky to find the power density spectrum of $v[n]$. This is beyond the scope of this class.

References

[1] A. Oppenheim, R. Schafer, and J. Buck, *Discrete-time Signal Processing.* Upper Saddle River, NJ: Prentice Hall, 2nd ed., 1999.