Problem Set 12

EECS123: Digital Signal Processing

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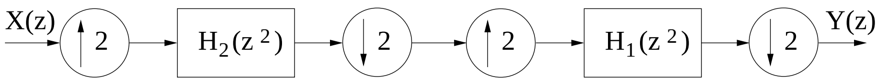
Important Note: Solutions of Problem 1 to 5 should be submitted by Monday, 05/05/08. Problem 6 to 9 are intended as study material for the exam. It will be assumed that you have gone through all the problems in this homework.

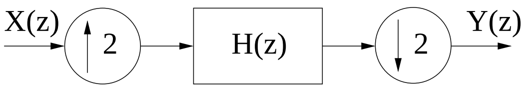
Solutions to all the problems will be provided by 05/05/08.

1. Multirate Identities

   1. Find the overall transfer function of the following system:



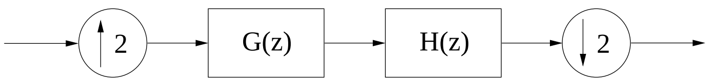
   2. In the system below, if $H(z) = E_0(z^2) + z^{-1}E_1(z^2)$, prove that $Y(z) = X(z)E_0(z)$.



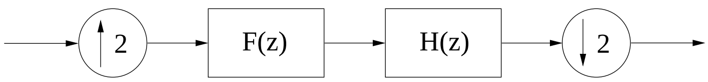
   3. Let $H(z), F(z)$ and $G(z)$ be filters satisfying

$$H(z)G(z) + H(-z)G(-z) = 2$$
$$H(z)F(z) + H(-z)F(-z) = 0.$$

   Prove that one of the following systems is unity and the other zero.

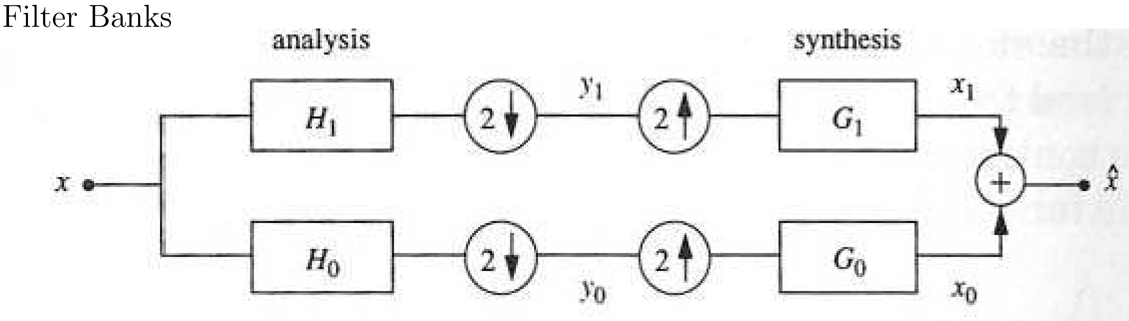


(a) system 1



(b) system 2

2. Filter Banks



The diagram above shows the analysis and synthesis structure of a simple filter bank.

(a) Let H_0 and H_1 be "brickwall" low-pass and high-pass filters. If z_0 and z_1 are the outputs of the analysis filter bank when the input is y_0 (the output of H_0 in the diagram), draw the simplified form of the filter structure between x and z_1. Hint: combine filters and downsamplers. This is known as a two-stage decomposition.

(b) Sketch the frequency response of the filter in your simplified structure.

(c) What downsampling factor is used in your simplified structure?

3. We will now implement the analysis and synthesis filter bank (single stage) shown in the diagram above in MATLAB.

(a) First we need to design approximate H_0 and H_1 brickwall filters. Using fir1() (window method), design 10th order (length 11) filters that approximate the complementary brickwall filters for our filter bank.

(b) Now we will analyze and reconstruct a simple signal using three types of analysis filters. Using each specific filter type below, determine y_0, y_1 (the decomposition coefficients needed to reconstruct the signal). Then reconstruct the signal using the synthesis filters. Be aware that each analysis and synthesis filter has a delay that you need to compensate for. Also, due to the filter length, the convolution of the input and filter impulse response will be longer than it should be (causing the reconstructed signal to be longer).

Use as an input $x = [3,\ 3,\ 1,\ 2,\ 3,\ 4,\ 5]$.

i. Approximate Brickwall Filters
Each filter has a delay of 5, so compensate for this by throwing out the first 5 values of $y_1,\ y_0,\ x_1,\ x_2$.

ii. Haar Wavelet
The Haar wavelet basis is $g_0 = \frac{1}{\sqrt{2}}[1,\ \ 1]$, $g_1 = \frac{1}{\sqrt{2}}[1,\ \ -1]$. The analysis filters have a delay of 1, the synthesis filters have no delay.

iii. **Optional:** Daubechies 4-Tap Wavelet
This wavelet has a basis:
$g_0 = \frac{\sqrt{2}}{8}[1+\sqrt{3},\quad 3+\sqrt{3},\quad 3-\sqrt{3},\quad 1-\sqrt{3}]$,
$g_1 = \frac{\sqrt{2}}{8}[1-\sqrt{3},\quad -(3-\sqrt{3}),\quad 3+\sqrt{3},\quad -(1+\sqrt{3})]$
The total system has a delay of 3 that needs to be compensated for in the final output.

(c) Is the reconstruction perfect for each pair of filters? Explain why or why not.

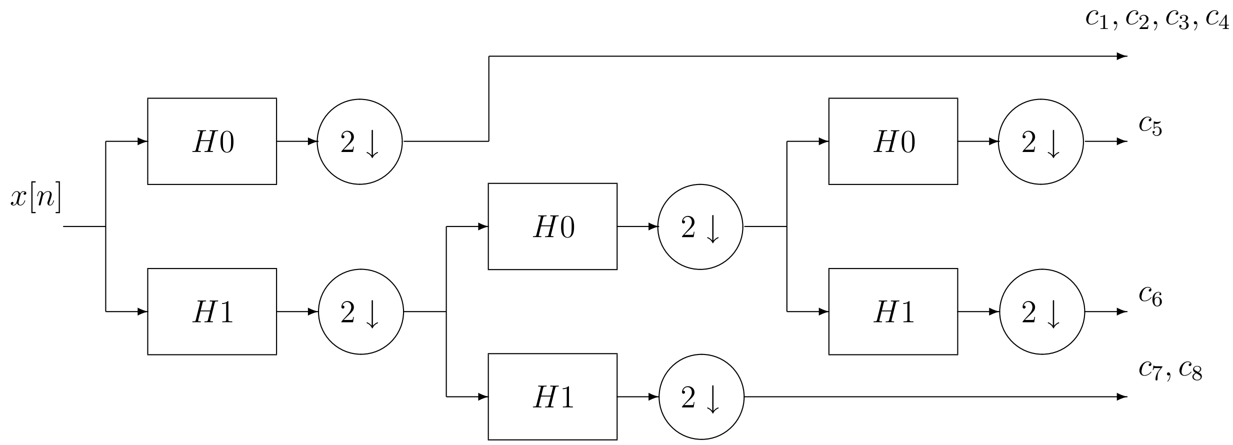
(d) Using the sample speech signal from the website `speech_fe.wav`, decompose the signal using the analysis filters and then reconstruct the signal using the synthesis filters (for each of the three pairs). Be sure to compensate for any delay. Calculate the mean-square error between each reconstructed signal and the original signal. Listen to the reconstructed waveforms and comment on your observations.

4. For $M = 4$ channels, we want the four polyphase components of $H(z)$, where,

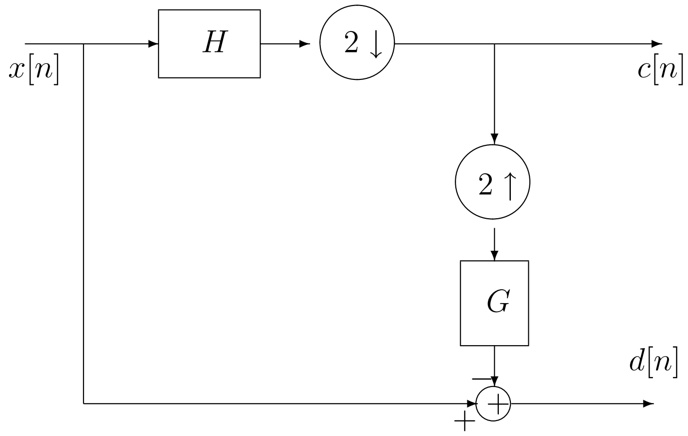
$$H(z) = \frac{1}{1 - az^{-1}}.$$

If $H(z) = H_0(z^4) + z^{-1}H_1(z^4) + z^{-2}H_2(z^4) + z^{-3}H_3(z^4)$, then find $H_i(z), i = 0, 1, 2, 3$.

5. For the following Haar filter bank structure, sketch the time-frequency tiling. Assume $x[n]$ is a length 8 signal. Label your sketch with the appropriate coefficients as shown in the diagram.



6. **Optional:** The structure shown below is called a Laplacian Pyramid. It is similar to the analysis filter banks we have been studying. An important difference is if we use a length N signal as input, the output at $c[n]$ is length $N/2$, and the output at $d[n]$ is length N. Therefore, we are keeping $3N/2$ values, more than we started with.



   (a) Draw the synthesis stage that takes $c[n]$ and $d[n]$ and reconstructs the original signal $x[n]$. You can assume that H and G are ideal lowpass filters.

   (b) What happens if H and G are approximated ideal lowpass filters (Problem 4). Implement the analysis and synthesis stages in Matlab to investigate your hypothesis.

   (c) What can we do to $d[n]$ in order to create another output signal of length $N/2$ and still be able to reconstruct the original signal from this new length $N/2$ signal and $c[n]$?

      i. Show what must be done to $d[n]$ if H and G are ideal lowpass filters with cutoff $\pi/2$.

      ii. Do the same if H and G are the Haar lowpass analysis and synthesis filters repecitively.

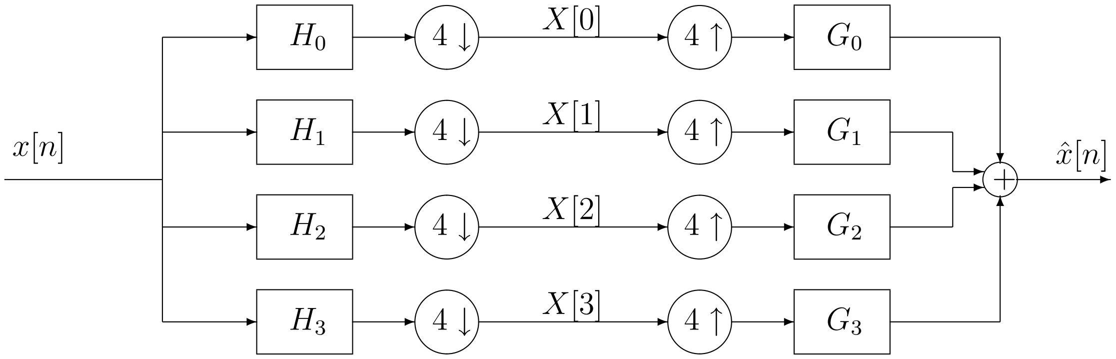
7. **Optional:** Let $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 4z^{-4} + 3z^{-5} + 2z^{-6} + z^{-7}$. Let $H(z) = H_{\text{even}}(z^2) + z^{-1}H_{\text{odd}}(z^2)$. Find the polyphase components $H_{\text{even}}(z)$ and $H_{\text{odd}}(z)$.

   In general, what is the relation between $H_{\text{even}}(z)$ and $H_{\text{odd}}(z)$ for antisymmetric filters of even length and symmetric filters of odd length.

8. **Optional:** The Normalized DFT of a length N sequence is defined as follows:

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

   We wish to compute the Normalized DFT $\{X[0], X[1], X[2], X[3]\}$ of a length 4 sequence using the 4 channel filter bank shown below.



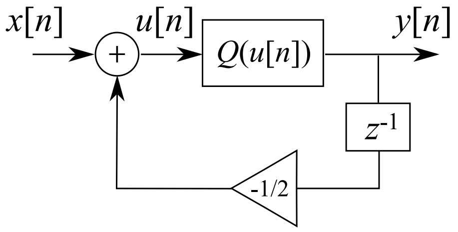
(a) Find the analysis filters $\{h_i[n]\}_{i=0}^{3}$ and synthesis filters $\{g_i[n]\}_{i=0}^{3}$ used to implement this filter bank. If the analysis filters are to be made causal, what is the delay introduced by the system?

(b) Implement the filter bank in MATLAB. Use $x = [3, 4, 2, 1]$ as an input. Show that the DFT of the length 4 input signal is calculated correctly by the analysis filter bank (compare with the output of fft()). Show that the synthesis filter bank correctly reconstructs the original signal from the output of the analysis bank.

(c) If the input to the filter bank is a length 16 signal, what does the output of the analysis filter bank represent? Confirm your hypothesis in MATLAB.

9. **Optional:** Any number can be written in the sign-magnitude representation. Let x be any number over a finite dynamic range $[-1, 1]$. Then x can be written as $\pm \cdot b_1 b_2 b_3 \ldots$, where $b_1, b_2, b_3 \ldots$ are binary digits and $\pm$ represents the sign of x. This representation is called as the sign-magnitude representation.

In any practical circuit, a finite number of bits is used to represent any signal. Let B be the number of bits used. One bit will be used for the sign of $x[n]$. Other $B - 1$ bits represent the magnitude of $x[n]$ in a digital filter. In this problem, we will restrict $B = 3$. Let $x[n]$ be an input to a filter as shown in the figure below:



Quantization is non-linear and it is described next. There are eight points in the interval $[-1, 1]$, called (quantization) levels, $L := \{\frac{-7}{8}, \frac{-5}{8}, \frac{-3}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}$. The quantizer $Q(x)$ maps x to the closest point in L. Note that the points in L can be represented using 3-bits.

Question: Within this setup, for $x[n] = 0, n \geq 0$ and $y[-1] = 0.5$, compute the output $y[n]$ for all $n \geq 0$.

Remarks: The correct $y[n]$ exhibits a limit cycle (periodic oscillation). This periodic oscillation in a stable filter, for zero input, is due to quantization or finite-precision representation effects.