1 Introduction

The Discrete Fourier Transform (DFT) is the discrete time, discrete frequency analog of the continuous time fourier transform. One of its main uses is for frequency domain analysis and filtering on digital computers. (Another common application is filtering using discrete time convolution $y[n] = x[n] * h[n]$). One of the main advantages of the DFT is that it can be calculated in $O(n \log n)$ time (using the Fast Fourier Transform), while a discrete time convolution takes $O(n^2)$.

2 Derivation of DFT

Computers represent discrete rather than continuous signals of finite duration. Figure 1 shows how a finite length signal $x(t)$ can be manipulated to obtain a discrete time/ discrete frequency approximation. (That is, we have a sampled time and sampled frequency representation of the original signal) Consider some time signal $x(t)$ with continuous time fourier transform $X(j\omega)$. If we sample $x(t)$ in time we have a periodic spectrum (DTFT) of $X_\delta(j\omega) = X(e^{j\omega T_s})$. How do we obtain a discrete spectrum which the computer can represent? We know that a periodic-in-time signal will have a discrete spectrum. Thus convolution of the time signal with a comb function (making the time signal periodic) will make the spectrum discrete. Figure 1 shows this process: first multiply by a comb to obtain a discrete-time/continuous-periodic-spectrum, then convolve with a comb to obtain DT-periodic/DF-periodic.

Summary:

i) Uniform samples in time implies periodic spectrum.

ii) Periodic time function implies uniformly spaced discrete spectrum.

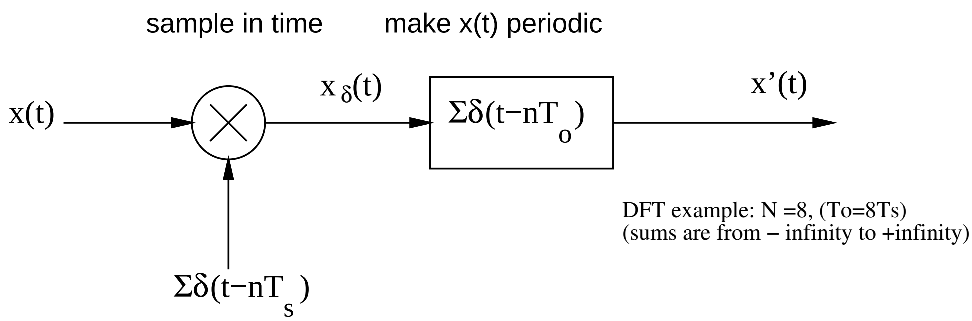
iii) Combining i) and ii) yields periodic and sampled in time with periodic and sampled in frequency.

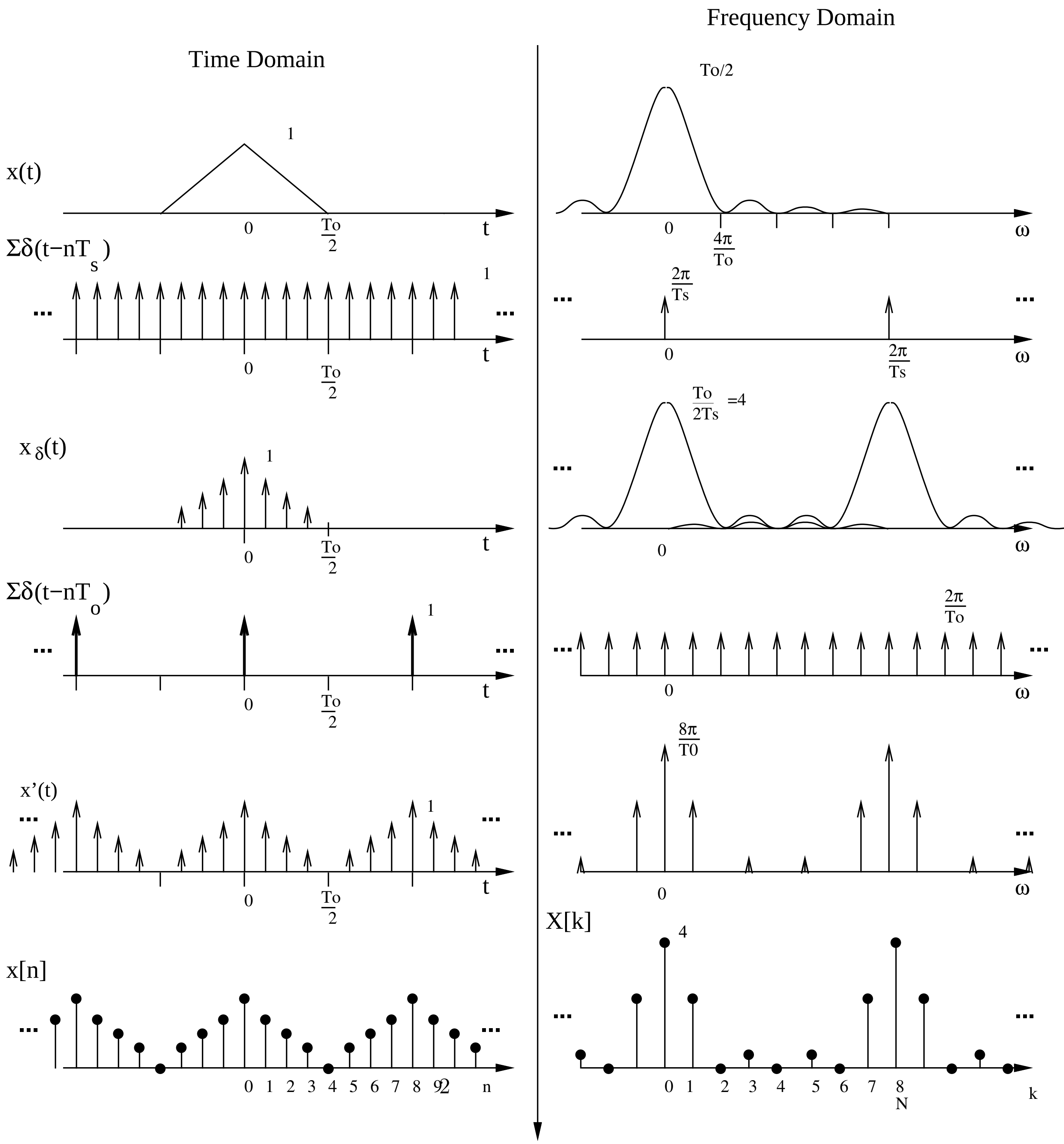
Now we will see mathematically the derivation of the DFT. A continuous time signal $x(t)$ with duration less than T_o is sampled in time at rate $T_s = \frac{T_o}{N}$ where T_s is the sampling rate and N is the total number of samples. If $x(t)$ is not finite duration, it must be windowed in time. Note that $x(t)$ can not be strictly bandlimited. (Why?)

$$x_\delta(t) = \sum_{n=0}^{N-1} x(nT_s)\delta(t - nT_s)$$

The spectrum is:

$$\begin{aligned} X_\delta(j\omega) &= \sum_{n=0}^{N-1} x(nT_s)e^{-j\omega nT_s} \\ &= X(e^{j\omega T_s}) \\ &= \sum_{n=0}^{N-1} x(\frac{nT_o}{N})e^{\frac{-j\omega nT_o}{N}} \end{aligned}$$


sample in time
make x(t) periodic
x(t)
x δ(t)
Σδ(t−nT o)
x'(t)
Σδ(t−nT s)
DFT example: N =8, (To=8Ts)
(sums are from − infinity to +infinity)


Time Domain
Frequency Domain
x(t)
Σδ(t−nT s)
x δ(t)
Σδ(t−nT o)
x'(t)
x[n]
X[k]
To/2
4π/To
2π/Ts
To/2Ts =4
2π/To
8π/T0
t
ω
n
k
N

Now convolve $x_\delta(t)$ with a comb to make it periodic with period T_o:

$$x'(t) \;=\; x_\delta(t) * \sum_{n\;=\;-\infty}^{\infty} \delta(t - nT_o)$$

The spectrum is then:

$$\begin{aligned}
X'(j\omega) \;&=\; X_\delta(j\omega) \cdot \frac{2\pi}{T_o} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{k2\pi}{T_o}) \\
&=\; \frac{2\pi}{T_o} \sum_{k=-\infty}^{\infty} X_\delta(\frac{jk2\pi}{T_o})\delta(\omega - \frac{k2\pi}{T_o}) \\
&=\; \frac{2\pi}{T_o} \sum_{k=-\infty}^{\infty} \left(\sum_{n=0}^{N-1} x(\frac{nT_o}{N}) e^{\frac{-j2\pi knT_o}{T_o N}} \right) \; \delta(\omega - \frac{k2\pi}{T_o}) \\
&=\; \frac{2\pi}{T_o} \sum_{k=-\infty}^{\infty} \left(\sum_{n=0}^{N-1} x(\frac{nT_o}{N}) e^{\frac{-j2\pi kn}{N}} \right) \; \delta(\omega - \frac{k2\pi}{T_o})
\end{aligned}$$

Observations:

1) $X'(j\omega)$ is discrete in frequency, with frequency sample spacing of $\Delta\omega \;=\; \frac{2\pi}{T_o}$.

2) $X'(j\omega)$ is periodic in frequency with period $\frac{2\pi}{T_s} \;=\; \frac{2\pi N}{T_o}$. (Note that $X_\delta(j\omega) \;=\; X_\delta(\omega \;+\; \frac{2\pi N}{T_o})$.

Now lets change the notation slightly to get rid of the impulses in frequency. In the new notation, the discrete variable has an amplitude equal to the area of the corresponding impulse. Define $x[n]$ as:

$$x[n] \;=\; x(nT_s) \;=\; x(\frac{nT_o}{N})$$

Lets define $X[k]$ (the Discrete Fourier Transform of $x[n]$) as:

$$X[k] \;=\; \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi kn}{N}} \;=\; \sum_{n=0}^{N-1} x[n] {W_N}^{nk}$$

where $W_N \;=\; e^{\frac{-j2\pi}{N}}$. The index k is referred to as the *frequency index*, and takes on integer values $0, 1, 2, ...N-1$. Then by sifting

$$X'(\frac{m2\pi}{T_o}) \;=\; \frac{2\pi}{T_o} \sum_{k=-\infty}^{\infty} X[k]\delta(\frac{m2\pi}{T_o} - \frac{k2\pi}{T_o}) \;=\; \frac{2\pi}{T_o} X[m]\delta(0)$$

Note that there is a simple relation between the DFT and $X'(j\omega)$:

$$X[k] \;=\; \frac{T_o}{2\pi} \text{area}[X'(jk\frac{2\pi}{T_o})]$$

This relation is particularly useful to be able to calculate the DFT indirectly, without having to evaluate the DFT sum. (For N large, this is a useful shortcut for manual calculation).

3 DFT Relations

Of course, the DTFS from Lecture 3 is just about the same thing as the DFT, except for the placement of the $\frac{1}{N}$ factor. For compatability with Matlab, we will use the DFT form shown here:

Discrete Fourier Transform

$$X[k] \;=\; \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi kn}{N}}$$

Inverse DFT

$$x[n] \;=\; \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{\frac{j2\pi kn}{N}}$$

The DFT spectrum is periodic: $X[k] \;=\; X[k+N]$.

Duality

$$DFT\left\{\frac{1}{N} X[k]\right\} \;=\; x[-n]$$

Central Ordinate

$$x[0] \;=\; \frac{1}{N} \sum_{k=0}^{N-1} X[k] \quad \text{and} \quad X[0] \;=\; \sum_{n=0}^{N-1} x[n]$$