1. ***Causality:***

For a causal continuous-time LTI system, we have

$$h(t)=0 \qquad t<0$$

Since $h(t)$ is a right-sided signal, the corresponding requirement on $H(s)$ is that the ROC of $H(s)$ must be of the form

$$\operatorname{Re}(s)>\sigma_{\max}$$

That is, the ROC is the region in the s-plane to the right of all of the system poles. Similarly, if the system is anticausal, then

$$h(t)=0 \qquad t>0$$

and $h(t)$ is left-sided. Thus, the ROC of $H(s)$ must be of the form

$$\operatorname{Re}(s)<\sigma_{\min}$$

That is, the ROC is the region in the s-plane to the left of all of the system poles.

2. ***Stability:***

In Sec. 2.3 we stated that a continuous-time LTI system is BIBO stable if and only if [Eq. (*2.21*)]

$$\int_{-\infty}^{\infty}|h(t)|\,dt<\infty$$

The corresponding requirement on $H(s)$ is that the ROC of $H(s)$ contains the $j\omega$-axis (that is, $s=j\omega$) (Prob. 3.26).

3. ***Causal and Stable Systems:***

If the system is both causal and stable, then all the poles of $H(s)$ must lie in the left half of the s-plane; that is, they all have negative real parts because the ROC is of the form $\operatorname{Re}(s)>\sigma_{\max}$, and since the $j\omega$ axis is included in the ROC, we must have $\sigma_{\max}<0$.

C. System Function for LTI Systems Described by Linear Constant-Coefficient Differential Equations:

In Sec. 2.5 we considered a continuous-time LTI system for which input $x(t)$ and output $y(t)$ satisfy the general linear constant-coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}=\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{3.38}$$

Applying the Laplace transform and using the differentiation property (*3.20*) of the Laplace transform, we obtain

$$\sum_{k=0}^{N} a_k s^k Y(s)=\sum_{k=0}^{M} b_k s^k X(s)$$

In this case, the system function $H(s)$ is an improper fraction and can be rewritten as

$$H(s)=\frac{1}{R}\frac{s+1/RC-1/RC}{s+1/RC}=\frac{1}{R}-\frac{1}{R^2C}\frac{1}{s+1/RC}$$

Since the system is causal, taking the inverse Laplace transform of $H(s)$, the impulse response $h(t)$ is

$$h(t)=\mathscr{L}^{-1}\{H(s)\}=\frac{1}{R}\delta(t)-\frac{1}{R^2C}e^{-t/RC}u(t)$$

Note that we obtained different system functions depending on the different sets of input and output.

3.24. Using the Laplace transform, redo Prob. 2.5.

From Prob. 2.5 we have

$$h(t)=e^{-\alpha t}u(t) \qquad x(t)=e^{\alpha t}u(-t) \qquad \alpha>0$$

Using Table 3-1, we have

$$H(s)=\frac{1}{s+\alpha} \qquad \operatorname{Re}(s)>-\alpha$$

$$X(s)=-\frac{1}{s-\alpha} \qquad \operatorname{Re}(s)<\alpha$$

Thus,

$$Y(s)=X(s)H(s)=-\frac{1}{(s+\alpha)(s-\alpha)}=-\frac{1}{s^2-\alpha^2} \qquad -\alpha<\operatorname{Re}(s)<\alpha$$

and from Table 3-1 (or Prob. 3.6) the output is

$$y(t)=\frac{1}{2\alpha}e^{-\alpha|t|}$$

which is the same as Eq. (*2.67*).

3.25. The output $y(t)$ of a continuous-time LTI system is found to be $2e^{-3t}u(t)$ when the input $x(t)$ is $u(t)$.

(*a*) Find the impulse response $h(t)$ of the system.

(*b*) Find the output $y(t)$ when the input $x(t)$ is $e^{-t}u(t)$.

(*a*) $$x(t)=u(t),\ y(t)=2e^{-3t}u(t)$$

Taking the Laplace transforms of $x(t)$ and $y(t)$, we obtain

$$X(s)=\frac{1}{s} \qquad \operatorname{Re}(s)>0$$

$$Y(s)=\frac{2}{s+3} \qquad \operatorname{Re}(s)>-3$$

Hence, the system function $H(s)$ is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s}{s+3} \qquad \operatorname{Re}(s) > -3$$

Rewriting $H(s)$ as

$$H(s) = \frac{2s}{s+3} = \frac{2(s+3)-6}{s+3} = 2 - \frac{6}{s+3} \qquad \operatorname{Re}(s) > -3$$

and taking the inverse Laplace transform of $H(s)$, we have

$$h(t) = 2\delta(t) - 6e^{-3t}u(t)$$

Note that $h(t)$ is equal to the derivative of $2e^{-3t}u(t)$ which is the step response $s(t)$ of the system [see Eq. (*2.13*)].

(*b*)
$$x(t) = e^{-t}u(t) \longleftrightarrow \frac{1}{s+1} \qquad \operatorname{Re}(s) > -1$$

Thus,

$$Y(s) = X(s)H(s) = \frac{2s}{(s+1)(s+3)} \qquad \operatorname{Re}(s) > -1$$

Using partial-fraction expansions, we get

$$Y(s) = -\frac{1}{s+1} + \frac{3}{s+3}$$

Taking the inverse Laplace transform of $Y(s)$, we obtain

$$y(t) = (-e^{-t} + 3e^{-3t})u(t)$$

3.26. If a continuous-time LTI system is BIBO stable, then show that the ROC of its system function $H(s)$ must contain the imaginary axis, that is, $s = j\omega$.

A continuous-time LTI system is BIBO stable if and only if its impulse response $h(t)$ is absolutely integrable, that is [Eq. (*2.21*)],

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

By Eq. (*3.3*)

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\, dt$$

Let $s = j\omega$. Then

$$|H(j\omega)| = \left| \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\, dt \right| \le \int_{-\infty}^{\infty} |h(t)e^{-j\omega t}|\, dt = \int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Therefore, we see that if the system is stable, then $H(s)$ converges for $s = j\omega$. That is, for a stable continuous-time LTI system, the ROC of $H(s)$ must contain the imaginary axis $s = j\omega$.