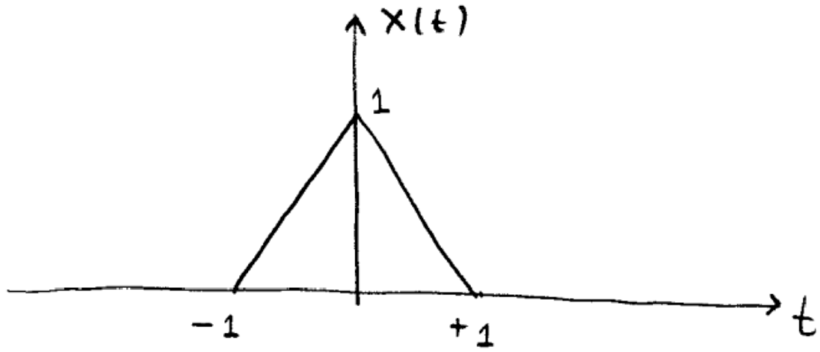
Discussion Section #4, 14 Feb 2014 Lillian Ratliff

Example 1. Consider the signal $x(t)$ as shown below

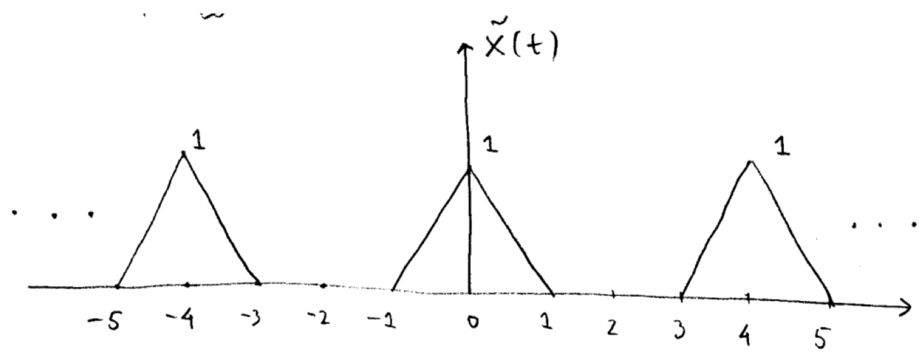


1. Find the FT $X(j\omega)$ of $x(t)$
2. Sketch the signal
$$\tilde{x}(t) = x(t) * \sum_{-\infty}^{\infty} \delta(t-4k)$$
3. Find another signal $g(t)$ such that $g(t)$ is not the same as $x(t)$ and
$$\tilde{x}(t) = g(t) * \sum_{k=-\infty}^{\infty} \delta(t-4k)$$

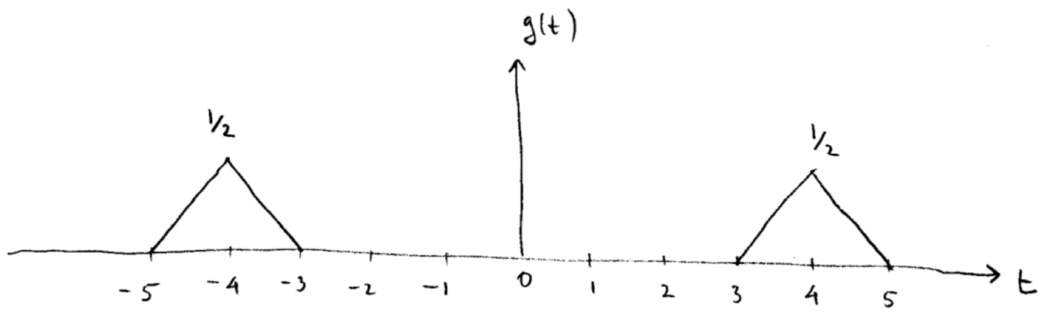
Solution:

1. Not that $x(t) = x_1(t) * x_1(t)$ where
$$x_1(t) = \begin{cases} 1, & |t| \leq \frac{1}{2} \\ 0, & \text{o.w.} \end{cases}$$
Also,
$$X_1(j\omega) = \int_{-1/2}^{1/2} e^{-j\omega t}\, dt = \frac{e^{-j\omega t}}{-j\omega}\Big|_{t=-1/2}^{1/2} = \frac{e^{-j\omega/2} - e^{j\omega/2}}{-j\omega} = \frac{2}{\omega}\frac{e^{j\omega/2} - e^{-j\omega/2}}{2j} = 2\frac{\sin\left(\frac{\omega}{2}\right)}{\omega}$$
Using the convolution property, we get
$$X(j\omega) = X_1(j\omega)X_1(j\omega) = \left(2\frac{\sin\left(\frac{\omega}{2}\right)}{\omega}\right)^2$$

2. $$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t-4k)$$



3. One possible choice for $g(t)$ is



Example 2. Find the DTFT of $na^{n+1}u[n+1]$, $|a| < 1$.

Solution:
Note that

$$na^{n+1}u[n+1] = (n+1)a^{n+1}u[n+1] - a^{n+1}u[n+1]$$

(This is just zero rewritten). Further,

$$a^n u[n] \longleftrightarrow \frac{1}{1 - ae^{-j\omega}}$$

The DTFT of the second term can be obtained by the time–shift property

$$a^{n+1}u[n+1] \longleftrightarrow \frac{e^{j\omega}}{1 - ae^{-j\omega}}$$

The DTFT of the first term can be obtained using the frequency differentiation property

$$na^n u[n] \longleftrightarrow j\frac{d}{d\omega}\left(\frac{1}{1 - ae^{-j\omega}}\right) = j(-1)\frac{-ae^{-j\omega}(-j)}{(1 - ae^{-j\omega})^2} = \frac{ae^{-j\omega}}{(1 - ae^{-j\omega})^2}$$

Hence,

$$(n+1)a^{n+1}u[n+1] \longleftrightarrow \frac{ae^{-j\omega}e^{j\omega}}{(1 - ae^{-j\omega})^2} = \frac{a}{(1 - ae^{-j\omega})^2}$$

Combining the two pieces we get

$$na^{n+1}u[n+1] \longleftrightarrow \frac{a}{(1 - ae^{-j\omega})^2} - \frac{e^{j\omega}}{1 - ae^{-j\omega}}$$

Example 3. Find the DTFT of $x[n] \equiv 1$

Solution:
Note that if $\tilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$ then

$$\tilde{x}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \; d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n} \; d\omega = \frac{1}{2\pi} \quad \forall n$$

Hence, if $x[n] \equiv 1$, we have

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Example 4. Consider the difference equation

$$y[n] - 1.2y[n-1] + 0.36y[n-2] = x[n] + x[n-1]$$

1. Find the frequency response $H(e^{j\omega})$
2. Use $H(e^{j\omega})$ to calculate the impulse response.

Solution:

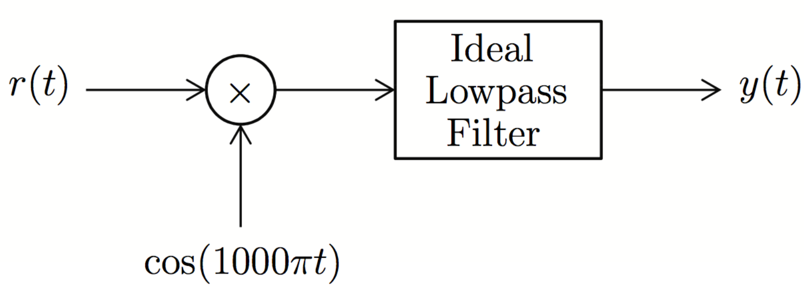
1.
$$h[n] - 1.2h[n-1] + 0.36h[n-2] = \delta[n] + \delta[n-1]$$
taking the FT we have
$$H(e^{j\omega}) \left(1 - 1.2e^{-j\omega} + 0.36e^{-j2\omega}\right) = (1 + e^{-j\omega})$$
so that
$$H(e^{j\omega}) = \frac{1 + e^{-j\omega}}{1 - 1.2e^{-j\omega} + 0.36e^{-j2\omega}}$$

2. Note that $z^2 - 1.2x + 0.36 = (z - 0.6)^2$ so we can use partial fraction expansion:
$$H(e^{-j\omega}) = \frac{A}{1 - 0.6e^{-j\omega}} + \frac{B}{(1 - 0.6e^{-j\omega})^2}$$
Hence,
$$A - 0.6Ae^{-j\omega} + B = 1 + e^{-j\omega} \implies A = -\frac{5}{3}, \; B = \frac{8}{3}$$
plugging in these values and taking the inverse FT we get
$$h[n] = -\frac{5}{3}(0.6)^n u[n] + \frac{8}{3}(n+1)(0.6)^n u[n] = (0.6^n + \frac{8}{3} n (0.6)^n) u[n]$$

Example 5. Let $S(t)$ be a real-valued signal for which $S(j\omega) = 0$ when $|\omega| > 1000\pi$. Amplitude modulation is performed to produce the signal:

$$r(t) = s(t)\sin(1000\pi t)$$

and the demodulation scheme depicted below is applied to $r(t)$ at the receiver end. Determine $y(t)$ assuming that the ideal lowpass filter has a cutoff frequency of 1000π and a passband gain of 1.



Solution:
Let us denote by $x(t)$ the signal that is filtered by the ideal low pass filter

$$x(t) = s(t)\sin(1000\pi t)\cos(1000\pi t) = \frac{1}{2}s(t)\sin(2000\pi t)$$

Taking the FT of $x(t)$ we get

$$X(j\omega) = \frac{1}{4j}(S(j(\omega - 2000\pi)) - S(j(\omega + 2000\pi)))$$

(since $\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$ and we know that $e^{j\theta}g(t) \longleftrightarrow G(\omega - \theta)$.) Since $s(t)$ is bandlimited to 1000π it is easy to conclude that $X(j\omega)$ is non-zero only in the range $[-3000\pi, -1000\pi]$ and $[1000\pi, 3000\pi]$. Because the low–pass filterpasses frequency in the range from -1000π to 1000π, $Y(j\omega) = 0$, and thus $y(t) = 0$.