Fourier Transform Properties and Examples

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Ex. FT of $e^{-\alpha t} u(t)$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-\alpha t} u(t) e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(\alpha + j\omega)t} dt = \frac{-1}{\alpha + j\omega} e^{-(\alpha + j\omega)t} \Big|_{0}^{\infty}$$

But, $|e^{-j\omega t}| = 1$. Thus as $t \to \infty$, $e^{-(\alpha + j\omega)t} = e^{-\alpha t} e^{-j\omega t} = \infty$ if $\alpha < 0$, but is equal to zero if $\alpha > 0$.

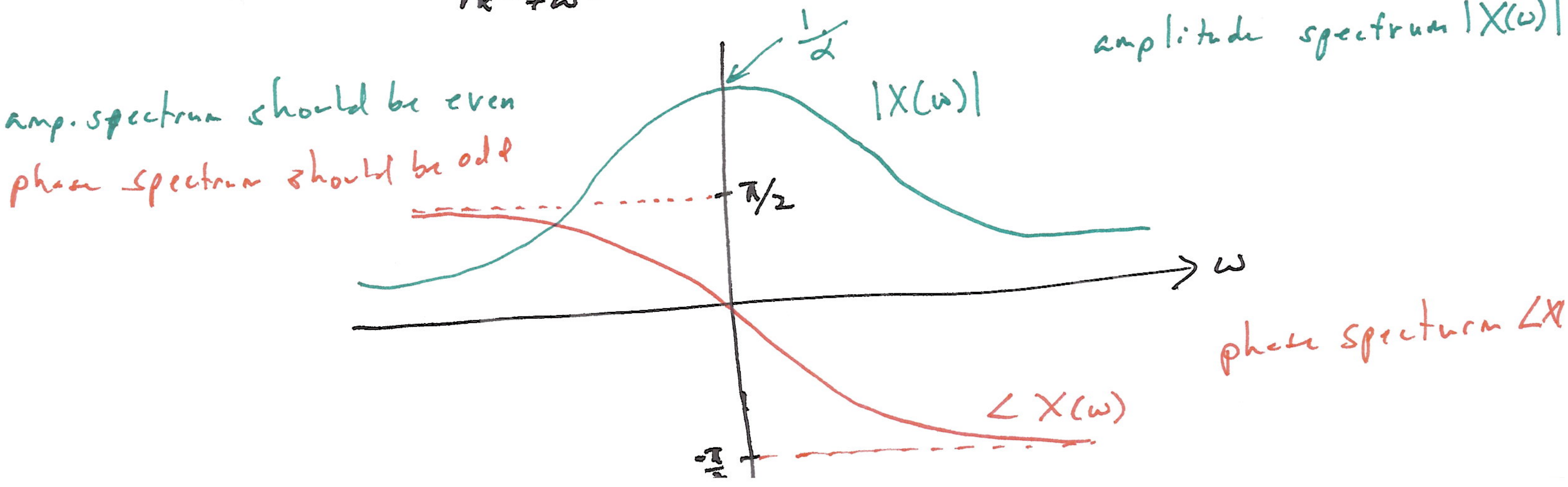
Thus,

$$\boxed{X(\omega) = \frac{1}{\alpha + j\omega}, \quad \alpha > 0} \qquad (1)$$

• $\alpha + j\omega \xrightarrow{\text{polar form}} \sqrt{\alpha^2 + \omega^2}\, e^{j \tan^{-1}(\omega/\alpha)}$ so that (1) becomes

$$\boxed{X(\omega) = \frac{1}{\sqrt{\alpha^2 + \omega^2}} e^{-j \tan^{-1}\left(\frac{\omega}{\alpha}\right)}}$$

Thus, $|X(\omega)| = \frac{1}{\sqrt{\alpha^2 + \omega^2}}$ and $\angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{\alpha}\right)$

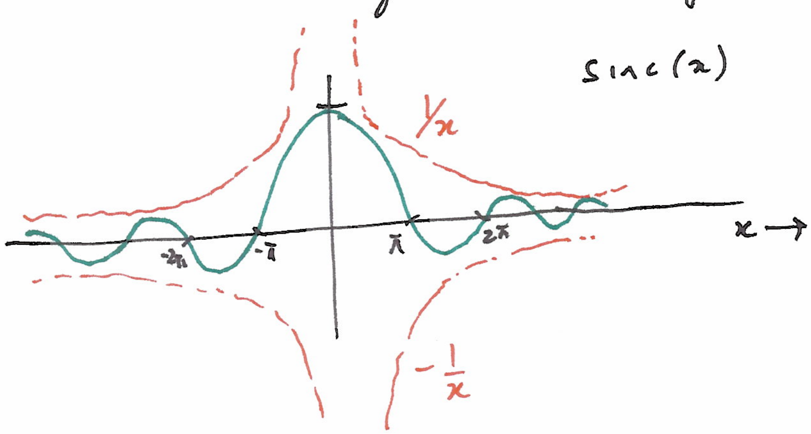


Sinc fn.

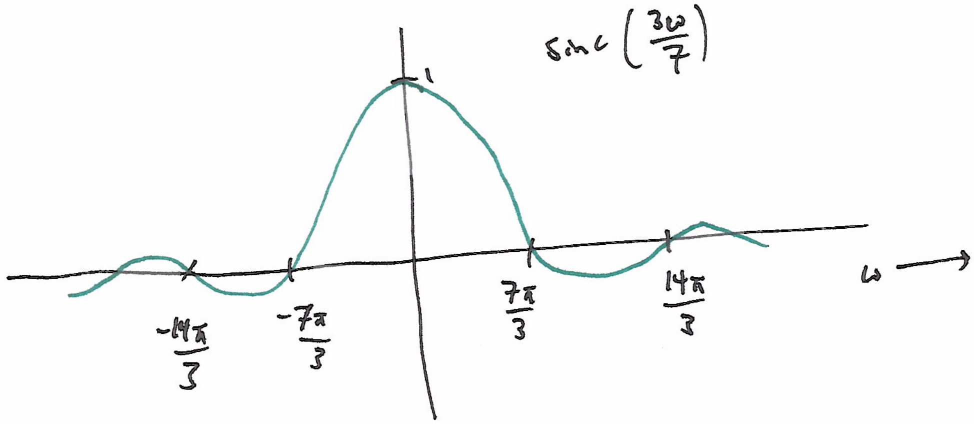
$$\text{Sinc}(x) = \frac{\sin(x)}{x}$$

1. $\text{sinc}(x) = 0$ when $\sin(x) = 0$ except at $x = 0$, where it appears to be indeterminate. i.e. $\text{sinc}(x) = 0$ for $x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$
2. $\text{sinc}(x)$ is an even fn. of x
3. by L'Hôpital's rule, $\text{sinc}(0) = 1$
4. $\text{Sinc}(x)$ is the product of an oscillating signal $\sin x$ of period 2π and a monotonically decreasing fn. $1/x$. Thus, $\text{sinc}(x)$ exhibits damped oscillations of period 2π w/ amplitude decreasing continuously as $\frac{1}{x}$.

e.g. 1)



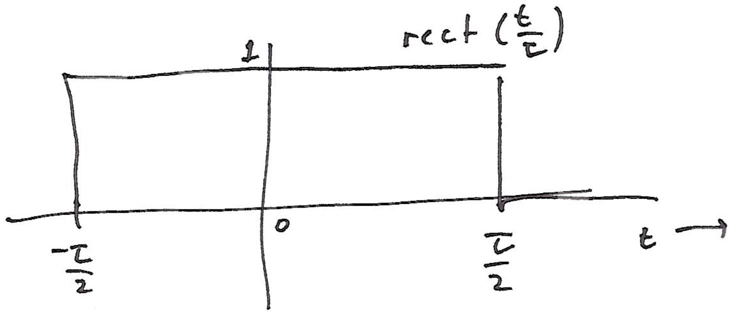
2)



Ex's ctd.

1) Find $\mathcal{F}\{x(t)\} = \mathcal{F}\{\text{rect}(\frac{t}{\tau})\}$

Note:

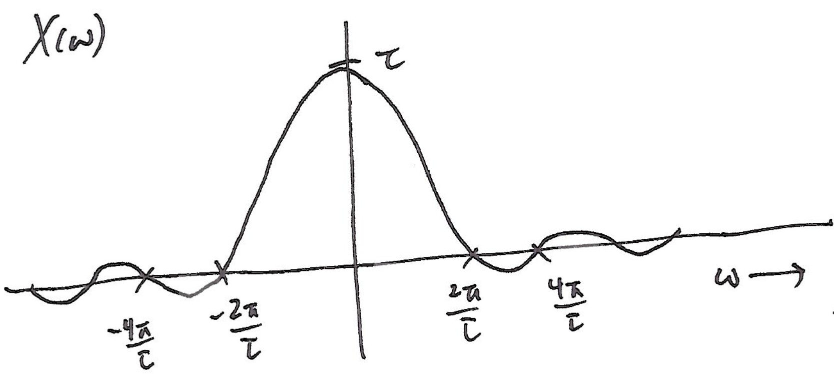


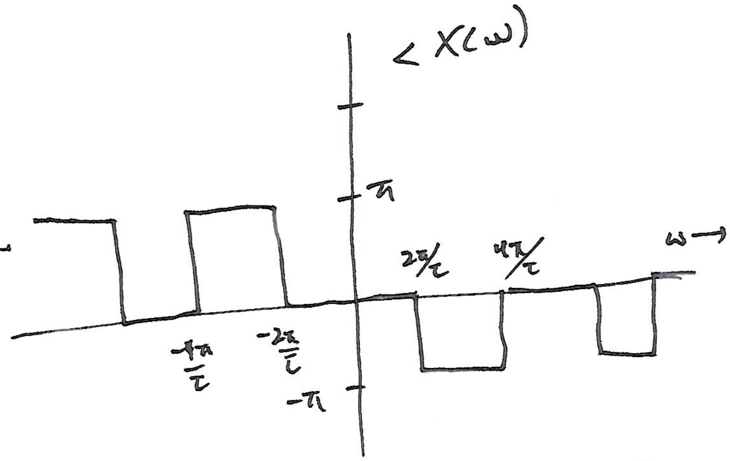
$$X(\omega) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t}\, dt$$

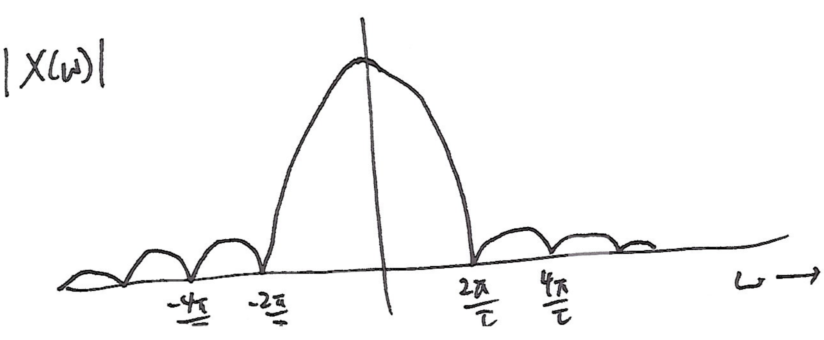
Since $\text{rect}(\frac{t}{\tau}) = 1$ for $|t| < \frac{\tau}{2}$ and since it is zero for $|t| > \frac{\tau}{2}$

$$X(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t}\, dt = -\frac{1}{j\omega}\left(e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$

$$= \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau \,\text{sinc}\left(\frac{\omega\tau}{2}\right)$$







Note: $\text{sinc}\left(\frac{\omega\tau}{2}\right) = 0$ when

$\omega\frac{\tau}{2} = \pm n\pi$ i.e. when

$\omega = \pm \frac{2n\pi}{\tau}$ $(n = 1, 2, 3 \ldots)$

Properties

- Fourier Transform is <u>linear</u>; that is, if

$$\mathcal{F}\{x_1(t)\} = X_1(\omega) \quad ; \quad \mathcal{F}\{x_2(t)\} = X_2(\omega)$$

then $\mathcal{F}\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 \mathcal{F}\{x_1(t)\} + a_2 \mathcal{F}\{x_2(t)\} = a_1 X_1(\omega) + a_2 X_2(\omega)$

<u>Time shifting property:</u>

if $\mathcal{F}\{x(t)\} = X(\omega)$ then $\cancel{\mathcal{F}\{x(t-t_0)\}}$

$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)e^{-j\omega t_0}$$

Dual w the <u>freq. shifting property</u>:

$$x(t)e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega-\omega_0)$$

- Conjugation Symmetry: $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \Rightarrow x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-\omega)$

- Duality: $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \Rightarrow X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$

- Scaling: $x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

- Time convolution: $x_1(t) * x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega) X_2(\omega)$

- Freq. convolution: $x_1(t) x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$

Ex. Freq. convolution

$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}$$

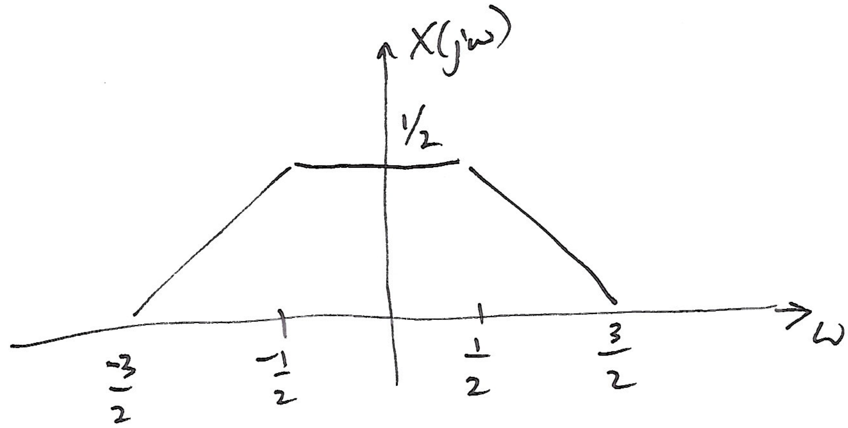
This can be written as the product of two sinc fns

$$x(t) = \pi\left(\frac{\sin(t)}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right)$$

$$\xrightarrow{MP} X(j\omega) = \frac{1}{2}\,\mathcal{F}\left\{\frac{\sin(t)}{\pi t}\right\} * \mathcal{F}\left\{\frac{\sin(t/2)}{\pi t}\right\}$$

We know that $\mathcal{F}\left\{\frac{\sin Wt}{\pi t}\right\} = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$

Hence we would convolve the resulting pulses & get



$$X(\omega) = |X(\omega)|e^{j\phi(\omega)}$$

Use properties:

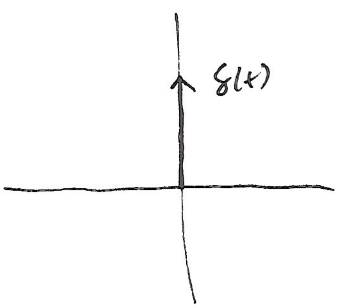
FT of $x(t) = 1$:

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = \underset{\Vert}{e^{-j\omega \cdot 0}} = 1 \qquad \star \quad \delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$$

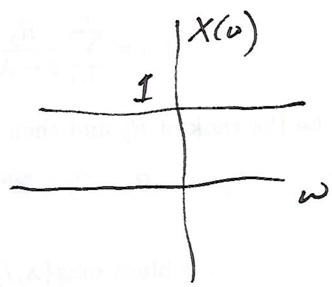
Then by the duality property

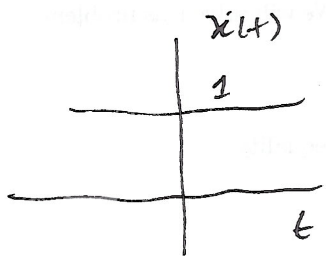
$$1 \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, \delta(-\omega) = 2\pi\, \delta(\omega) \quad \star$$

In pics.

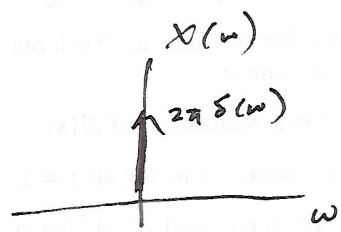


$\overset{\mathcal{F}}{\longleftrightarrow}$





$\overset{\mathcal{F}}{\longleftrightarrow}$



Now consider $x(t) = e^{+j\omega_0 t}$ what is FT?

Applying freq. property to $\star$ we get

$$e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, \delta(\omega - \omega_0)$$

What about $x(t) = e^{-j\omega_0 t}$

$$e^{-j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, \delta(\omega + \omega_0)$$

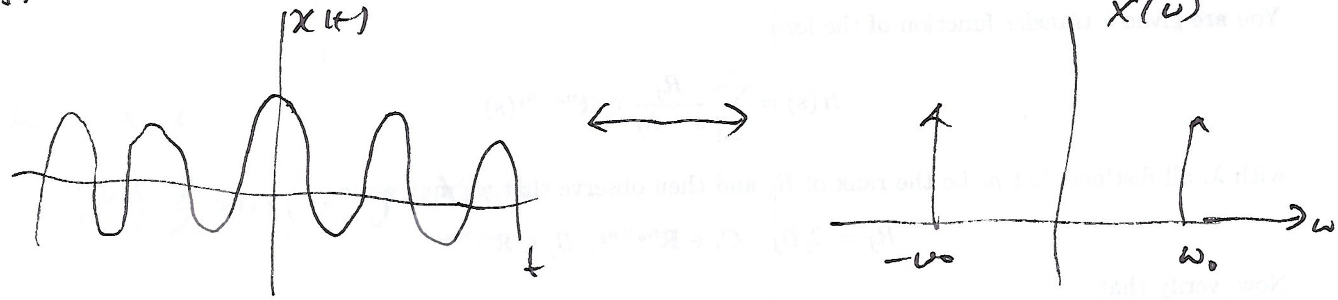
Now consider $x(t) = \cos\omega_0 t$

Euler: $\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$

using what we just derived, we get

$$\cos\omega_0 t \xleftrightarrow{\mathcal{F}} \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

Figs.



Now consider $z(t) = y(t)\cos\omega_0 t$

by freq. convolution ~~theorem~~ prop.,

$$y(t)\cos\omega_0 t \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} Y(\omega) * \left[\pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)\right]$$

$$= \frac{1}{2} Y(\omega-\omega_0) + \frac{1}{2} Y(\omega+\omega_0)$$

Consider the following ODE:

$$y'(t) + 2y(t) = x(t) + x'(t)$$

Taking FT we can get the freq. response much easier

$$\mathcal{F}\{y'(t) + 2y(t)\} = \mathcal{F}\{x(t) + x'(t)\}$$

Note that

$$\mathcal{F}\{\frac{d}{dt}x(t)\} = j\omega X(\omega)$$

So,

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega) + j\omega X(\omega)$$

$$\Rightarrow (j\omega + 2)Y(\omega) = (1 + j\omega)X(\omega)$$

$$\Rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1+j\omega}{2+j\omega} = \frac{2+j\omega-1}{2+j\omega} = 1 - \frac{1}{2+j\omega}$$