Discussion Section #2, 31 Jan 2014 Lillian Ratliff

1 Unit Impulse

The unit impulse (Dirac delta) has the following properties:

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)\ dt = 1$$

Remark 1. Important!: An ordinary function with is everywhere 0 except at a single point must have the integral 0 (in the Riemann integral sense). Thus, $\delta(t)$ cannot be an ordinary function and mathematically it is defined (in the weak sense) by

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)\ dt = \phi(0) \tag{TF}$$

where $\phi(t)$ is any regular function continuous at $t = 0$. Note that (TF) is a symbolic expression and should be considered an ordinary Riemann integral. In this sense, $\delta(t)$ is often called a *generalized* function (distribution) and $\phi(t)$ is known as a *test function*.

Definition 1 (Delayed Delta)**.**

$$\int_{-\infty}^{\infty} \phi(t)\delta(t - t_0)\ dt = \phi(t_0) \tag{DD}$$

2 Convolution

Definition 2 (Convolution Integral)**.**

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\ d\tau \tag{CI}$$

Example 1. Show the following properties:

1. $x(t) * \delta(t) = x(t)$
   Solution:
   By (DD) and (CI) we have

   $$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\ d\tau = x(\tau)\big|_{\tau=t} = x(t)$$

2. $x(t) * \delta(t - t_0) = x(t - t_0)$
   Solution:
   Since $*$ is commutative and by (DD),

   $$x(t) * \delta(t - t_0) = \delta(t - t_0) * x(t) = \int_{-\infty}^{\infty} \delta(\tau - t_0)x(t - \tau)\ d\tau = x(t - \tau)\big|_{\tau=t_0} = x(t - t_0)$$

3. $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\ d\tau$
   Solution:
   Since

   $$u(t - t_0) = \begin{cases} 1, & t > t_0 \\ 0, & t < t_0 \end{cases}$$

   we have

   $$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)u(t - \tau)\ d\tau = \int_{-\infty}^{t} x(\tau)\ d\tau$$

4. $x(t) * u(t - t_0) = \int_{-\infty}^{t-t_0} x(\tau)\ d\tau$
Solution:
Since

$$u(t - \tau - t_0) = \begin{cases} 1, & \tau < t - t_0 \\ 0, & \tau > t - t_0 \end{cases}$$

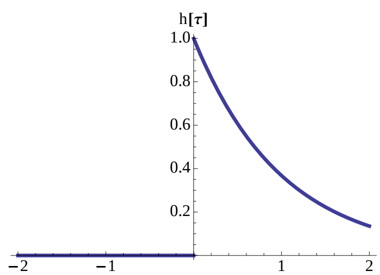
we have

$$x(t) * u(t - t_0) = \int_{-\infty}^{\infty} x(\tau) u(t - \tau - t_0)\ d\tau = \int_{-\infty}^{t-t_0} x(\tau)\ d\tau$$

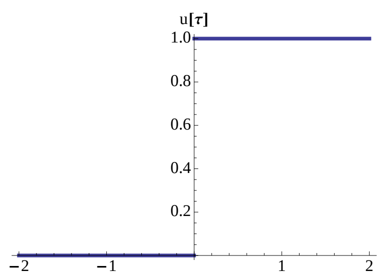
Example 2. Consider

$$x(t) = u(t) \quad \text{and} \quad h(t) = e^{-\alpha t} u(t),\ \alpha > 0$$

Compute $y(t)$ using (CI).
Solution:

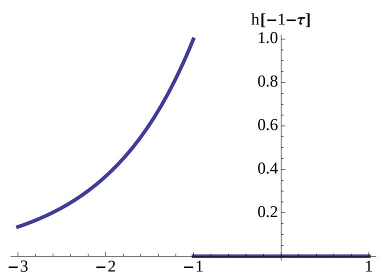


(a) Graph of $h(\tau)$

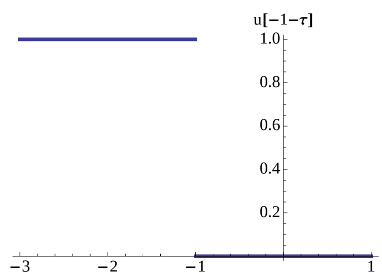


(b) Graph of $u(\tau)$

Figure 1: Graphs of Functions for Example 2

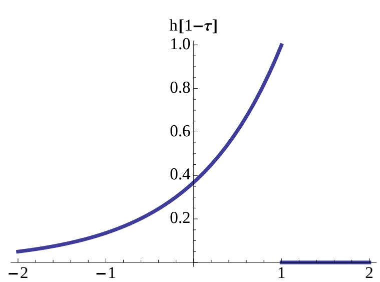


(a) Graph of $h(-1 - \tau)$

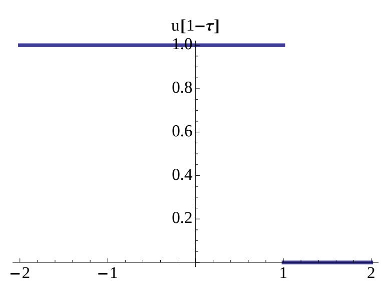


(b) Graph of $u(-1 - \tau)$

Figure 2: Graphs of Functions for Example 2



(a) Graph of $h(1 - \tau)$



(b) Graph of $u(1 - \tau)$

Figure 3: Graphs of Functions for Example 2

Functions $u(\tau)$ and $h(t-\tau)$ are shown in figures 2 and 3 for $t = 1 > 0$ and $t = -1 < 0$. For $t < 0$, $u(\tau)$ and $h(t-\tau)$ do not overlap, while for $t > 0$, they overlap from $\tau = 0$ to $\tau = t$. Hence, for $t < 0$, $y(t) = 0$.For $t > 0$, we have

$$y(t) = \int_0^t e^{-\alpha(t-\tau)}\, d\tau = e^{-\alpha t}\int_0^t e^{\alpha\tau}\, d\tau = e^{-\alpha t}\frac{1}{\alpha}(e^{\alpha t}-1) = \frac{1}{\alpha}(1-e^{-\alpha t})$$

Thus,

$$y(t) = \frac{1}{\alpha}(1-e^{-\alpha t})u(t)$$

Example 3.

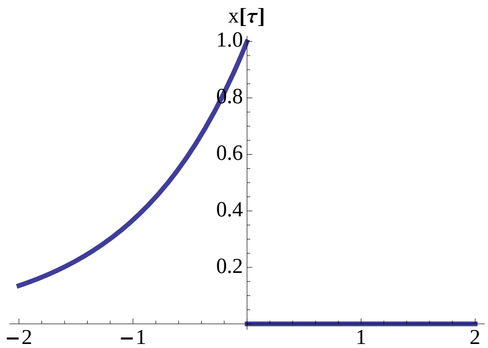


Figure 4: Example 3 graph of $x(\tau)$

(a) Graph of $h(1-\tau)$

(b) Graph of $h(-1-\tau)$

Figure 5: Graphs of Functions for Example 2

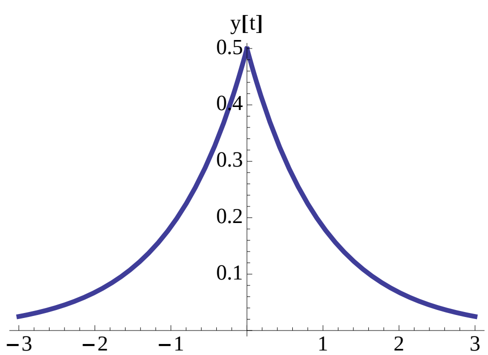


Figure 6: Example 3 graph of $y(t)$

Consider

$$h(t) = e^{-\alpha t}u(t) \quad \text{and} \quad x(t) = e^{\alpha t}u(-t),\ \alpha > 0$$

Figures 4 and 5 depict $x(\tau)$ and $h(t-\tau)$ for $t = 1, -1$. It is clear from the pictures that for $t < 0$, $x(\tau)$ and $h(t-\tau)$ overlap from $\tau = -\infty$ to $\tau = t$, while for $t > 0$ they overlap from $\tau = -\infty$ to $\tau = 0$. Hence, for $t < 0$, we have

$$y(t) = \int_{-\infty}^{t} e^{\alpha\tau} e^{-\alpha(t-\tau)}\ d\tau = e^{-\alpha t} \int_{-\infty}^{t} e^{2\alpha\tau}\ d\tau = \frac{1}{2\alpha} e^{\alpha t}$$

For $t > 0$ we have

$$y(t) = \int_{-\infty}^{0} e^{\alpha\tau} e^{-\alpha(t-\tau)}\ d\tau = e^{-\alpha t} \int_{-\infty}^{0} e^{2\alpha\tau}\ d\tau = \frac{1}{2\alpha} e^{-\alpha t}$$

Combining the above two cases we have

$$y(t) = \frac{1}{2\alpha} e^{-\alpha|t|},\ \alpha > 0$$

which is depicted in Figure 6.

3 Linear Differential Equation

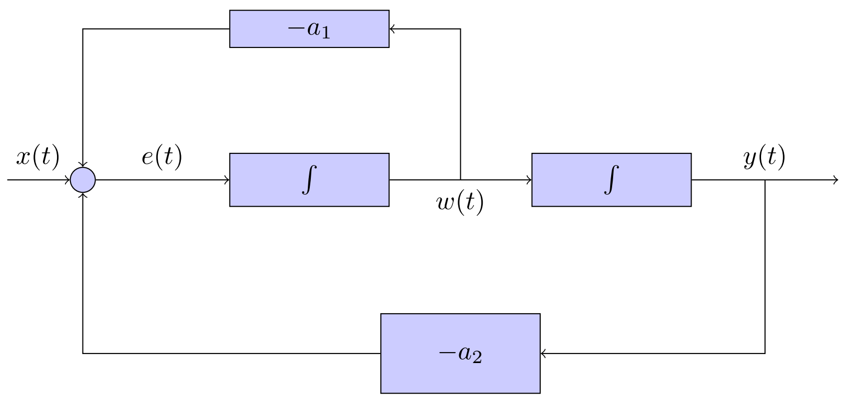


Figure 7: LDE example

Example 4. Write a differential equation that relates the output $y(t)$ and the input $x(t)$.

Solution:
Let $e(t)$ and $w(t)$ be the input and the output of the first integrator in Figure 7. Hence,

$$e(t) = \frac{dw(t)}{dt} = -a_1 w(t) - a_2 y(t) + x(t)$$

Since $w(t)$ is the input to the second integrator, we have

$$w(t) = \frac{dy(t)}{dt}$$

Hence,

$$\frac{d^2 y(t)}{dt^2} = -a_1 \frac{dy(t)}{dt} - a_2 y(t) + x(t)$$

Example 5. Consider the CT system whose input and output are related by

$$\frac{dy(t)}{dt} + a y(t) = x(t)$$

where a is constant. Find $y(t)$ with the initial condition $y(0) = y_0$ and with input

$$x(t) = K e^{-bt} u(t)$$

Solution:
Let $y(t) = y_p(t) + y_h(t)$ where $y_p(t)$ is the particular solution and $y_h(t)$ is the homogeneous solution, i.e. it satisfies

$$\frac{dy_h(t)}{dt} + ay_h(t) = 0 \tag{H}$$

Assume that $y_p(t) = Ae^{-bt},\ t > 0$. Substituting $y_p(t)$ into the given ODE, we have

$$-bAe^{-bt} + aAe^{-bt} = Ke^{-bt}$$

from which we get

$$A = \frac{K}{(a-b)} \quad \text{and} \quad y_p(t) = \frac{K}{a-b}e^{-bt},\ t > 0$$

To obtain $y_h(t)$ we assume

$$y_h(t) = Be^{st}$$

since the homogeneous solution will have this form. Substituting into (H) gives,

$$sBe^{st} + aBe^{st} = (s+a)Be^{st} = 0$$

so that $s = -a$ and $y_h(t) = Be^{-at}$. Thus,

$$y(t) = Be^{-at} + \frac{K}{a-b}e^{-bt}$$

Using the initial data we get

$$B = y_0 - \frac{K}{a-b}$$

Thus,

$$y(t) = \left(y_0 - \frac{K}{a-b}\right)e^{-at} + \frac{K}{a-b}e^{-bt},\ t > 0$$

For $t < 0$, we have $x(t) = 0$ so that the original ode is just the homogeneous ode in (H). Thus, $y(t) = Be^{-at}$ and using the initial data we get

$$y(t) = y_0 e^{-at},\ t < 0$$

4 LCCDE

Example 6. Consider a system S with input $x[n]$ and output $y[n]$ related according to the block diagram in Figure 8. The input $x[n]$ is multiplied by $e^{-j\omega_0 n}$ and the product is passed through a stable LTI system with impulse

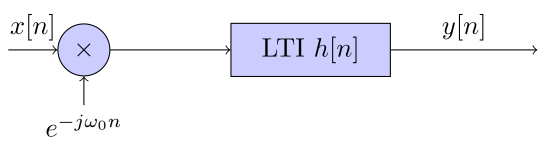


Figure 8: LDE example

response $h[n]$.

1. Is the system S linear?
   Solution:
   Yes.

$$y[n] = h[n] * (e^{-j\omega_0 n}x[n]) = \sum_{k=-\infty}^{\infty} e^{-j\omega_0 k}x[k]h[n-k]$$

Let $x[n] = ax_1[n] + bx_2[n]$, then

$$\begin{aligned}
y[n] &= h[n] * (e^{-j\omega_0 n}(ax_1[n] + bx_2[n])) \\
&= \sum_{k=-\infty}^{\infty} e^{-j\omega_0 k}(ax_1[n] + bx_2[n])h[n-k] \\
&= a\sum_{k=-\infty}^{\infty} e^{-j\omega_0 k}x_1[k]h[n-k] + b\sum_{k=-\infty}^{\infty} e^{-j\omega_0 k}x_2[k]h[n-k] \\
&= ay_1[n] + by_2[n]
\end{aligned}$$

2. Is the system S time invariant?
   Solution:
   Let $x_2[n] = x[n-n_0]$, then

$$\begin{aligned}
y_2[n] &= h[n] * (e^{-j\omega_0 n}x_2[n]) \\
&= \sum_{k=-\infty}^{\infty} e^{-j\omega_0 (n-k)}x_2[n-k]h[k] \\
&\sum_{k=-\infty}^{\infty} e^{-j\omega_0 (n-k)}x[n-n_0-k]h[k] \\
&\neq y[n-n_0]
\end{aligned}$$

   So it is **not** time invariant.

3. Is the system S stable?
   Solution:
   Since the magnitude of $e^{-j\omega_0 n}$ is always bounded by 1 and $h[n]$ is stable, a bounded input $x[n]$ will always produce a bounded input to the stable LTI system and therefore the output $y[n]$ will be bounded. Thus the system is stable.

4. Specify a system C such that the block diagram in Figure 9 represents an alternative way of expressing the input–output relationship of the system S.
   Solution:
   We can write $y[n]$ as follows:

$$\begin{aligned}
y[n] &= h[n] * (e^{-j\omega_0 n}x[n]) \\
&= \sum_{k=-\infty}^{\infty} e^{-j\omega_0 (n-k)}x[n-k]h[k] \\
&= \sum_{k=-\infty}^{\infty} e^{-j\omega_0 n}e^{j\omega_0 k}x[n-k]h[k] \\
&= e^{-j\omega_0 n}\sum_{k=-\infty}^{\infty} e^{j\omega_0 k}x[n-k]h[k]
\end{aligned}$$

   System C should therefore be a multiplication by $e^{-j\omega_0 n}$.

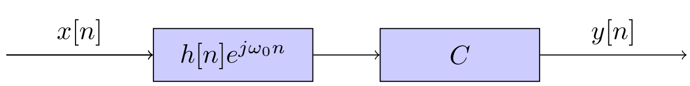


Figure 9: LDE example alternative

5 Fourier Series

Example 7. Determine the trigonometric Fourier series coefficients for a signal that is an impulse train with period T_0, i.e.

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

Solution:
Let

$$\delta_{T_0}(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)), \quad \omega_0 = \frac{2\pi}{T_0}$$

Since $\delta_{T_0}(t)$ is even, $b_k = 0$ and since

$$a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos(k\omega_0 t) \; dt$$

we have

$$a_k = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) \cos(k\omega_0 t) \; dt = \frac{2}{T_0}$$

Thus, we get

$$\delta_{T_0}(t) = \frac{1}{T_0} + \frac{2}{T_0} \sum_{k=1}^{\infty} \cos(k\omega_0 t), \quad \omega_0 = \frac{2\pi}{T_0}$$