Discussion Section #1, 24 Jan 2014

1 Review

- Top down approach

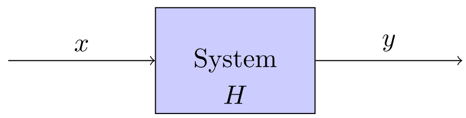

Figure 1: System

- Disk drive
- medical imaging
- control, elevator
- speech synthesis

2 Properties of LTI Systems

- System with and without memory
- Invertibility
- Causality
- Stability
- Time invariance
- Linearity: scaling and homogeniety

2.1 System with and without memory

Definition 1. A system is said to be *memoryless* if its output for each value of the independent variable at a given time is dependent only on the input at that same time.

Example 1. 1. Consider

$$y[n] = (2x[n] - x^2[n])^2.$$

Q: is it memoryless? why or why not?
A: yes, since the value of $y[n]$ at any particular time n_0 depends only on the value of $x[n]$ at that time.

2. Q: Is a resistor memoryless? What is the system model?
A: Let $x(t)$ be the input and it represents the current and let $y(t)$ be the output and it represents the voltage. Then, if R is the resistance, the I-O relationship is

$$y(t) = Rx(t).$$

It is a memoryless system.

identity system is memoryless.

$$y(t) = x(t) \quad \text{(CT)} \qquad y[n] = x[n] \quad \text{(DT)}$$

Example 2 (Systems with memory). 1. *accumulator* or *summer*

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

2. one–second *delay*

$$y[n] = x[n-1]$$

3. Capacitor: let C be the capacitance, the input be the current and the output be the voltage, then

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\, d\tau$$

Remark 1. Conceptually, the concept of memory in a system corresponds to the presence of a mechanism in the system that retains or stores information about input values at times other than the current time.

2.2 Invertibility

Definition 2. A system is said to be *invertible* if distinct inputs lead to distinct output. Conceptually, the idea is that system A is invertible if there exists system B such that when A is followed by system B the combined system is the identity system. (draw a picture)

Example 3. $y(t) = 2x(t)$, inverse is $w(t) = \frac{1}{2}y(t)$.

2.3 Causality

Definition 3. A system is *causal* if the output at any time depends only on values of the input at the present time and in the past. Another term for this property is *nonanticipative*.

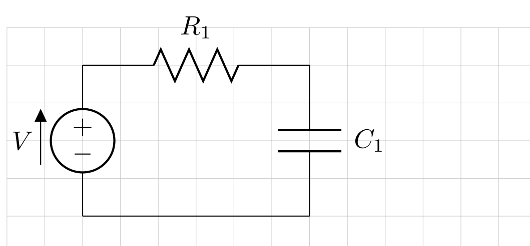

Figure 2: RC Circuit

Example 4. 1. The RC circuit in Figure 2 is causal since the capacitor voltage responds only to the present and past values of the source voltage.

2. The motion of an automobile is cause since it does not anticipate future actions of the driver (however this may change in the near future and in some cars already exists).

3. $y[n] = x[n] - x[n+1]$ and $y(t) = x(t+1)$ are NOT causal.

2.4 Stability

Definition 4. Informally, a *stable* system is one in which small inputs lead to responses that do not diverge (blow up).

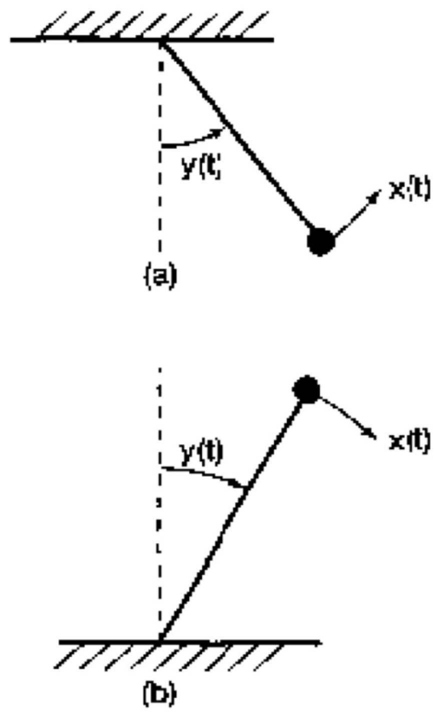

Figure 3: Pendulum

Example 5. 1. Pendulum: The input is the applied force $x(t)$ and the output is the angular deviation $y(t)$ from the vertical. Gravity applies a restoring force that tends to return the pendulum to the vertical position and frictional losses due to drag tend to slow it down. Thus, a small force $x(t)$ applied results in a deflection from vertical that is small.

2. Inverted pendulum: what happens here? Gravity tends to increase the deviation from vertical. So it becomes unstable.

Definition 5 (BIBO)**.** If the input to a *stable* system is *bounded* (magnitude does not grow without bound), then the output must also be bounded and therefore cannot diverge. Formally, if for any bounded input x defined by

$$|x| \leq k_1$$

the corresponding output y is also bounded defined by

$$|y| \leq k_2$$

where $k_1, k_2 < \infty$ are real constants, then the system is BIBO stable.

Example 6. The system in Figure 4 is a unit delay system. Is it stable?

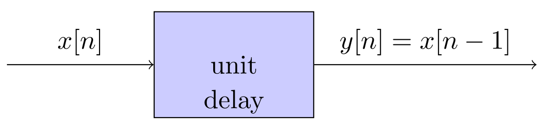

Figure 4: System

Yes, the system is BIBO stably since

$$|y[n]| = |x[n-1]| \leq k, \quad \text{if} \;\; |x[n]| \leq k \;\; \forall \; n$$

Example 7. Consider the system

$$y[n] = nx[n]$$

Is it BIBO stable?

No, it is not. Consider the following input as a counterexample: $x[n] = u[n]$. Then, $y[n] = nu[n]$ so that the bounded unit step sequence produces an output sequence that grows without bound.

2.4.1 Impulse Response

Recall that the impulse response to a system T is the response to an input $\delta(t)$:

$$h(t) = T\{\delta(t)\}$$

Definition 6. For an CT–LTI system, the condition for BIBO stability is that the impulse response be absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |h(t)| \; dt = \|h\|_1 < \infty$$

Example 8 (Verification of impulse response BIBO stability condition)**.** Assume that the input $x(t)$ of a CT–LTI system is bounded:

$$|x(t)| \leq k_1 \quad \forall \; t$$

Then, using

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau) \; dt$$

we have

$$\begin{aligned}
|y(t)| &= \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau) \; d\tau \right| \\
&\leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)| \; d\tau \\
&= \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)| \; d\tau \\
&\leq k_1 \int_{-\infty}^{\infty} |h(\tau)| \; d\tau
\end{aligned}$$

where the last inequality holds since it was assumed that $|x(t)| \leq k_1$, $\forall \; t$. Thus, if the impulse response is absolutely integrable (has finite L_1–norm), i.e.

$$\int_{-\infty}^{\infty} |h(\tau)| \; d\tau = K < \infty$$

then

$$|y(t)| \leq k_1 K = k_2$$

and the system is BIBO stable.

Example 9 (Examples of computing impulse response and checking BIBO stability)**.** Consider two systems whose impulse responses are $h_1(t) = e^{-2t}u(t)$ and $h_2(t) = 2e^{-t}u(t)$ respectively and they are connected *in cascade* as pictured in Figure 5.

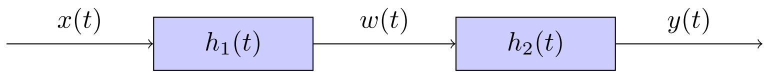

Figure 5: Cascade System

- Determine the impulse response $h(t)$ for the overall system

  Let $w(t) = x(t) * h_1(t)$. Then,
  $$y(t) = w(t) * h_2(t) = [x(t) * h_1] * h_2(t)$$
  Since convolution is associative, we can rewrite the above as
  $$y(t) = x(t) * [h_1(t) * h_2(t)] = x(t) * h(t)$$
  Therefore the impulse response for the overall system is $h(t) = h_1(t) * h_2(t)$.
  $$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} h_1(\tau)h_2(t-\tau)\ d\tau \\ &= \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)2e^{-(t-\tau)}u(t-\tau)\ d\tau \\ &= 2e^{-t}\int_{-\infty}^{\infty} e^{-\tau}u(\tau)u(t-\tau)\ d\tau \\ &= 2e^{-t}\left(\int_0^t e^{-\tau}\ d\tau\right)u(t) \\ &= 2(e^{-t} - e^{-2t})u(t) \end{aligned}$$

- Determine if the system is BIBO.

  Using the above $h(t)$, we have
  $$\int_{-\infty}^{\infty} |h(\tau)|\ d\tau = 2\int_0^{\infty}(e^{-\tau} - e^{-2\tau})\ d\tau = 2\left(\int_0^{\infty} e^{-\tau}\ d\tau - \int_0^{\infty} e^{-2\tau}\ d\tau\right) = 2(1 - \frac{1}{2}) = 1 < \infty$$
  Thus the system is BIBO.

2.5 Time–Invariance (Shift–Invariance)

Definition 7. A system is *time invariant* if a time shift in the input signal results in an identical time shift in the output signal. That is, if $y[n]$ is the output of a discrete–time, time–invariant system when $x[n]$ is the input, then $y[n-n_0]$ is the output when $x[n-n_0]$ is applied. In continuous time, if $y(t)$ is the output for a system when $x(t)$ is the input. Then, the system is time–invariant if $y(t-t_0)$ is the output when $x(t-t_0)$ is the input.

Example 10.
1. $y(t) = \sin(x(t))$ is time–invariant
2. $y[n] = nx[n]$ is time–varying

2.6 Linearity

Definition 8. A system is *linear* if it satisfies the following:

1. (*additivity*) The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
2. (*scaling* or *homogeneity*) The response to $ax(t)$ is $ay(t)$ where $a \in \mathbb{C}$.

Remark 2. We can combine these two properties into one single check:

$$ax_1(t) + bx_2(t) \longrightarrow ay_1(t) + by_2(t) \tag{CT}$$

$$ax_1[n] + bx_2[n] \longrightarrow ay_1[n] + by_2[n] \tag{DT}$$

where $a, b \in \mathbb{C}$.

Definition 9 (Superposition Property)**.**

$$x[n] = \sum_k a_k x_k[n] \longrightarrow y[n] = \sum_k a_k y_k[n]$$

Remark 3. A consequence of the superposition property is that, for linear systems, an input which is zero for all time results in an output which is zero for all time.

$$0 = 0 \cdot x[n] \longrightarrow 0 \cdot y[n] = 0$$

Example 11.

1. $y(t) = tx(t)$ is linear
2. $y(t) = x^2(t)$ is not linear
3. $y[n] = 2x[n] + 3$ is not linear (it is affine)