Notes 25 largely plagiarized by %khc

This material is optional. It will be a preview for those of you who wish to take ee123.
This is all massively plagiarized from Prof. Fearing's DFT handout from fall 1994.

1 Motivation

Given a discrete-time signal $x[n]$, we know that its Fourier transform can be obtained by taking its DTFT, or equivalently by taking its Z transform and evaluating that transform on the unit circle to obtain the DTFT. Unfortunately, the DTFT is continous-valued, and since there are an infinite number of points in one period of the DTFT, we cannot represent every point in that period using a computer.

The next best thing we can do is to sample the DTFT at specific points. However, sampling something in one domain forces its equivalent in the other domain to be discrete. The discrete Fourier transform (DFT) is the result.

To summarize, there are a number of reasons why we would wish to study the DFT:

- it gives us a way of finding the DTFT using a computer; this allows us to perform the spectral analysis of signals.
- there exists a computationally-inexpensive algorithm called the fast Fourier transform (FFT) that can compute the DFT; the FFT is $O(n \ln n)$, compared to the straightward $O(n^2)$ way of evaluating the DFT.
- having the FFT allows us to perform long convolutions relatively inexpensively; it turns out that if $y[n] = h[n] * x[n]$, the DFT of $y[n]$ is equal to the product of the DFTs of $h[n]$ and $x[n]$, subject to certain conditions.
- being able to perform convolutions quickly allows us to determine the output of digital filters with long inputs and long impulse responses in $O(n \ln n)$ time, instead of $O(n^2)$ using the standard convolution sum.

2 Discrete Time Transform (DFT)

To derive the DFT, consider the setup in Figure 1. Starting from $x(t)$ that is nonzero only over the interval from $t = 0$

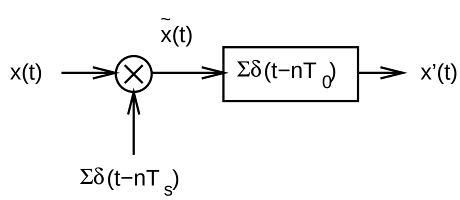

Figure 1: Interpreting the DFT.

to $t < T_0$, we sample $x(t)$ using a spacing between samples of $T_s = \frac{T_0}{N}$. We accomplish this by multiplying $x(t)$ by an impulse train $\sum_{n=-\infty}^{\infty} \delta(t - nT_s)$, giving $\tilde{x}(t)$:

$$\begin{aligned}
\tilde{x}(t) &= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \\
&= \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s) \\
&= \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \\
&= \sum_{n=0}^{N-1} x[n]\delta(t - nT_s)
\end{aligned}$$

Note that the limits of summation are reduced because $x(t)$ is of finite duration.

Then we pass $\tilde{x}(t)$ through a system with impulse response $\sum_{n=-\infty}^{\infty} \delta(t - nT_0)$, giving $x'(t)$; with T_0 chosen as an integer multiple of T_s [eg $T_0 = NT_s$], this makes $x'(t)$ periodic with period T_0:

$$\begin{aligned} x'(t) &= \tilde{x}(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \\ &= \sum_{n=-\infty}^{\infty} \tilde{x}(t - nT_0) \end{aligned}$$

What happens in the frequency domain? Once again, consider the setup in Figure 1. We start with $X(f)$, and convolve it with the Fourier transform of $\sum_{n=-\infty}^{\infty} \delta(t - nT_s)$, which is $\frac{1}{T_s}\sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{T_s})$; this gives $\tilde{X}(f)$:

$$\begin{aligned} \tilde{X}(f) &= X(f) * \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{T_s}) \\ &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(f - \frac{k}{T_s}) \end{aligned}$$

This is apparently a dead end, but from discussion in previous lectures, we recognize this as the DTFT:

$$\begin{aligned} \tilde{X}(f) &= X(e^{j2\pi f T_s}) \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n T_s} \\ &= \sum_{n=0}^{N-1} x[n] e^{-j2\pi f n T_s} \end{aligned}$$

Note that the limits of summation are reduced because $x(t)$ is of finite duration.

Now, since we convolved in the time domain to get $x'(t)$, we end up multiplying $\tilde{X}(f)$ by the Fourier transform of $\sum_{n=-\infty}^{\infty} \delta(t - nT_0)$, which is $\frac{1}{T_0}\sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{T_0})$. This effectively samples the DTFT in the frequency domain.

$$\begin{aligned} X'(f) &= \tilde{X}(f) \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{T_0}) \\ &= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \tilde{X}(f) \delta(f - \frac{k}{T_0}) \\ &= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \tilde{X}(\frac{k}{T_0}) \delta(f - \frac{k}{T_0}) \end{aligned}$$

Let's substitute the expression that we derived previously for $\tilde{X}(f)$:

$$\begin{aligned} X'(f) &= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \tilde{X}(\frac{k}{T_0}) \delta(f - \frac{k}{T_0}) \\ &= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} [\sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{T_0} n T_s}] \delta(f - \frac{k}{T_0}) \end{aligned}$$

Since $T_0 = NT_s$, we can substitute that in also:

$$\begin{aligned} X'(f) &= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} [\sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{T_0} n \frac{T_0}{N}}] \delta(f - \frac{k}{T_0}) \\ &= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} [\sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi n k}{N}}] \delta(f - \frac{k}{T_0}) \end{aligned}$$

Before going on, one should verify that:

- $\tilde{x}(t)$ is the continous-time representation of the discrete-time signal $x[n]$, with sampling period T_s.
- the corresponding $\tilde{X}(f)$ is just another name for $X(e^{j2\pi f T_s})$, the DTFT of $x[n]$.
- $\tilde{X}(f)$ is periodic with period $\frac{1}{T_s}$ [evaluate $\tilde{X}(f + \frac{1}{T_s})$].
- $x'(t)$ is periodic with period T_0.
- $x'(t)$ is discrete in time, with T_s between samples.
- $X'(f)$ is periodic with period $\frac{1}{T_s}$ [really not obvious; this arises out of the periodicity of $\tilde{X}(f)$].
- $X'(f)$ is discrete in frequency, with $\frac{1}{T_0}$ between samples.

If *any* one of these points is unclear, you are *strongly* encouraged to go back, reread lecture notes, and consult other references until they are clear.

Anyway, this is kind of cumbersome, so let's just lose the impulses in $X'(f)$. Define $X[k]$ equal to T_0 times the area of the impulse at $f = \frac{k}{T_0}$:

$$X[k] = T_0 X'(f)|_{f=\frac{k}{T_0}}$$

This then lets us define the discrete Fourier transform (DFT) as the quantity inside the square brackets in the last equation for $X'(f)$:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi nk}{N}}$$

Since everybody gets a little tired of writing that last complex exponential out, define $W_N = e^{-\frac{j2\pi}{N}}$.

Note that the k which indexes $X[k]$ is sometimes called the frequency index, and that its value ranges from 0 to $N - 1$.

3 Summary

The bottom line is this:

- DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

  with $k = 0, 1, \ldots, N - 1$.
- inverse DFT:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-nk}$$

  with $n = 0, 1, \ldots, N - 1$.
- $x[n]$ is clearly discrete.
- $x[n]$ is periodic with period N.
- $X[k]$ is clearly discrete.
- $X[k]$ is periodic with period N.
- The $X[k]$ are samples of $X(e^{j2\pi f T_s})$, the DTFT of $x[n]$; $X(e^{j2\pi f T_s})$ is sampled uniformly with spacing $\frac{1}{T_0}$.
- $k = 0$ corresponds to $f = 0$, $k = 1$ corresponds to $f = \frac{1}{T_0}$, and so on, up to $k = N - 1$ corresponding to $f = \frac{N-1}{T_0}$.